THESIS
ON
MULTIPLE HYPERGEOMETRIC FUNCTIONS AND THEIR APPLICATIONS IN STATISTICS
PRESENTED

FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN MATHEMATICS
OF

BUNDELKHAND UNIVERSITY, JHANSI (U.P.)
INDIA

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1992

CERTIFICATE

This is to certify that Mr. Pramod Kumar Vishwakarma actually carried out the work described in this thesis under my supervision at D.V. Postgraduate College, Orai (Jalaun). He has put the required attendance in the department during the period of his investigations.

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PREFACE

The present work is the out come of the further developments and extensions of researches done in the field of Multiple Hypergeometric Functions by me , under the research project " Multiple Hypergeometric Functions and their applications in Statistics " sponsored by Council of Science and Technology, U.P. Lucknow . , India. at D.V. Postgraduate College, Orai, U.P. India.

The thesis includes nine chapters. Every chapter is divided into several sections (Progressively numbered as 1.1, 1.2 , ... ,) , The formulae are numbered progressively within each section, e.g. , 6.3.9 refers to equation (9) of section 3 in chapter VI . References of the research works of other mathematicians are given in an alphabetical order at the end of every chapter.

First, I wish to express my deepest and sincerest feelings of graditude to Dr. R.C. Singh Chandel, M.Sc. Ph.D., Department of Mathematics, D.V. Postgraduate College Orai, U.P. , under whose kind supervision this thesis is being submitted , but for whose worthy guidance and encouragement it would not have been possible for me to accomplish my purpose.

I like to express my sincere thanks to Council of Science and Technology, Uttar Pradesh, Lucknow , and the authorities of Postgraduate College Orai for providing the necessary facilities to me during my present research work.

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LIST OF PUBLICATIONS

1. Karlsson's multiple hypergeometric function and its applications , Jñānābha ,19 (1989) , 173 - 185 .

2. Fractional integration and integral representations of Karlsson's multiple hypergeometric function and its confluent forms, Jñānäbha 20 (1990) , 101 - 110 .

3. Fractional derivatives of confluent hypergeometric forms of Karlsson's multiple hypergeometric function $^{(k)}F^{(n)}_{CD}$, Pure and Appl. Math. Sci. 35 , No, 1-2,(1992).

4. Fractional derivatives of the certain hypergeometric functions of several variables, (Presented in 49th Annual conference of National Academy of Sciences, India held at Osmania University Hyderabad on Nobember 26-28 , 1989) Also accepted for publication).

5. Multidimensional fractional derivatives of the multiple hypergeometric functions of several variables (Accepted for publication) .

6. Some relations between hypergeometric functions of three and four variables (Under communication) .

7. Generating relations for Multiple Hypergeometric Functions of several variables (Under communication) .

8. Fractional derivatives of the certain hypergeometric of four variables (Under communication) .

9. Some more confluent forms of multiple hypergeometric functions of several variables (Under communication) .

10. Fractional integration and integral representations of new confluent forms of multiple hypergeometric functions of several variables (Under communication) .

11. Fractional derivatives of new confluent multiple hypergeometric functions of several variables (Under communication) .

12. Some expectations associated with Beta and Gamma distributions involving some multiple hypergeometric function of Srivastava and Daoust (Under communication) .

13. Some expectations associated with multivariate Gamma and Beta distributions involving the multiple hypergeometric functions of Several Variables . (Under communication) .

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CONTENTS

INTRODUCTION

CHAPTER I

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INTRODUCTION

In the present chapter, we give a brief historical account of the work done so far in the field of " Multiple Hypergeometric Functions and Their Applications in Statistics ". No effort has been made to reproduce the complete and up-to-date history of the subject but only those points which have a direct connection with our work have been dealt with in some detail .

1.1 Hypergeometric Functions of One Variable.

The Gaussian Hypergeometric Series. In the study of second order linear differntial equations with three regular singular points, there arises the function

$$(1.1.1)\qquad {}_2F_1[a,b;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n}{(c)_n} \frac{z^n}{n!} \quad , c \neq 0 \ , -1,-2,\ldots$$

The above infinite series obviously reduces to the elementary geometric series

$$(1.1.2)\qquad \sum_{n=0}^{\infty} z^n = 1+z+z^2+\ldots$$

in the special cases when

(1.1.3) (i) $a = c$ and $b = 1$; (ii) $a = 1$, and $b = c$.

Hence it is called the hypergeometric series or more precisely, Gauss's hypergeometric series after the famous German mathematician Carl Friedrich Gauss (1777 – 1855) , who in the year 1812 introduced this series into analysis and gave the F – notation for it .

By D' Alembert ratio test, it is easily seen that the hypergeometric series in (1.1.1) converges absolutely within the unit circle, that is, when $|z|<1$, provided that the denominator parameter c is neither zero nor negative integer. If either or both of the numerator parameters a and b in (1.1.1) is zero or negative integer , the hypergeometric series terminates and therefore, the series is atomatically convergent.

Also series ${}_2F_1$ in (1.1.1) , when $|z| = 1$ (i.e. on the unit circle) is

(i) absolutely convergent if $Re(c-a-b) > 0$,

(ii) conditionally convergent if $-1 < Re(c-a-b) \leq 0$, $z \neq 1$

(iii) divergent if $Re(c-a-b) \leq -1$.

In case (i) we are led to the well known Gauss's summation theorem :

(1.1.4) $${}_2F_1[a,b;c;1] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} , \quad Re(c-a-b)>0 ,$$

$c = 0, -1, -2, \ldots$

As an its special case we have the summation formula

$$(1.1.5) \qquad {}_2F_1 [-n,b;c;1] = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, \ldots, \quad c \neq 0, -1, -2, \ldots,$$

which is identically eqivalent to Vandermode's convolution theorm :

$$(1.1.6) \qquad \sum_{k=0}^{n} \binom{\lambda}{k}\binom{\mu}{n-k} = \binom{\lambda+\mu}{n}, \quad n \geq 0 ,$$

λ, μ being any complex numbers .

For a number of summation theorems for the hypergeometric series (1.1.1) , when z takes on other special values one can refer to Bailey [3, pp. 9 - 11] , Erdélyi et al. [19, pp. 104 - 105] , Slater [86, p. 243] and Luke [48, pp 271 - 273] .

Generalized Hypergeometric Series . The natural generalization of above Gaussian hypergeometric series ${}_2F_1$ is given by the series

$$(1.1.7) \qquad {}_pF_q \left[\begin{matrix} a_1, \ldots, a_p ; \\ b_1, \ldots, b_q ; \end{matrix} \; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}$$

$$= {}_pF_q [a_1, \ldots, a_p; b_1, \ldots, b_q; z],$$

which is called generalized Gauss series, or simply, the generalized hypergeometric series . Here p,q are positive integers or

zero (interpreting an empty product as 1) , and we assume that the variable z , and parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ take on complex values , provided that

(1.1.8) $$b_j \neq 0, -1, -2, \ldots ; \; j = 1, \ldots, q .$$

If any parameter of numerator is zero or negative integer then ${}_pF_q$ will be terminating series i.e. it will be atomatically convergent , if no parameter of numerator is neither zero nor negative integer and (1.1.8) holds, then the series ${}_pF_q$ in (1.1.7)

(i) converges for $|z| < \infty$ if $p \leq q$

(ii) converges for $|z| < 1$ if $p = q+1$ and

(iii) diverges for all z , $z \neq 0$, if $p > q + 1$

Further more , if we take

(1.1.9) $$w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$$

then the series ${}_pF_q$, for $p = q + 1$, is

(I) absolutely convergent for $|z| = 1$ if $Re(w) > 0$,

(II) conditionally convergent for $|z| = 1$, $z \neq 1$, if $-1 < Re(w) \leq 0$ and

(III) divergent for $|z| = 1$ if $Re(w) \leq -1$.

An interesting further generalization of the series ${}_pF_q$ was given by Fox [29] and Wright ([110], [111]), who studied asymptotic expansion of the generalized hypergeometric function ${}_p\Psi_q$.

1.2 <u>Hypergeometric Series in two Variables</u>. The success of the theory of hypergeometric series in one variable has stimulated the development of a corresponding theory in two or more variables. Appell [1, p. 296] has defined the four double hypergeometric series F_1, F_2, F_3 and F_4 (known as Appell series) analogous to Gauss's ${}_2F_1[a,b;c;z]$.

The standard work on the theory of Appell series is the monograph by Appell and Kampé de Fériet [2], which contains an extensive bibliography of all relevant papers upto 1926 (by, for example, L. Pochhammer, J. Horn, É. Picard, É. Goursat). See Erdélyi et al. [19, pp. 222 - 245] for a review of a subsequent work on the subject; see also Bailey [3, chapter 9], Slater [86, chapter 8] and Exton [27, pp. 23 - 28] .

Horn puts

$$f(m,n) = \frac{F(m,n)}{F'(m,n)} , \qquad g(m,n) = \frac{G(m,n)}{G'(m,n)} ,$$

where F, F', G, G' are polynomials in m,n of respective degree p, p', q, q'. F' is assumed to have a factor $m+1$, and G' a factor $n+1$; F and F' have no common factor except possibily $n+1$. The greatest of the four numbers p, p', q, q', is the order of the hypergeometric series of order two and found that, apart from certain series which are either expressible in terms of one variable or are products of two hypergeometric series, each in one variable, there are essentially thirty four convergent series of order two (Horn [32] corrections in Brorngässer [5]) .

<u>Horn Series</u> . Horn [32] defined ten hypergeometric series in two variables and denoted them by G_1 , G_2 , G_3 , H_1 , ... , H_7 ; he thus completed the set of all possible second order (complete) hypergeometric series in two variables of Appell and Kampé de Fériet [2] (see also Erdélyi et al. [19 , pp. 224 - 228]).

<u>Confluent Hypergeometric Series in Two Variables</u>. Seven confluent forms of the four Appell series were defined by Humbert [33] and denoted by. Φ_1 , Φ_2 , Φ_3 , Ψ_1 , Ψ_2 ,

Ξ_1 , Ξ_2 .

In addition to above, there exist thirteen more confluent forms of the Horn series denoted by (Horn [32] and Borngässer [5]) :

Γ_1 , Γ_2 , H_1 , ... , H_{11} .

The work of Humbert has been described reasonable fully by Appell and Kampé de Fériet [2, pp. 124 - 135] , and the definitions and convergence conditions of all of these twenty confluent hypergeometric series in two variables are given also in Erdélyi et al. [19, pp 225 - 223] . The definitions of Φ_1 , Φ_2 and Ξ_2 , given in Erdélyi et al. [op. cit. p. 225, equations (20), (21) , and p. 226 equation (26)] are in error, Here we recall the corrected definition of Φ_1 and Φ_2 (the confluent forms of Appell function F_1) and for the sake of brevity, we delet the definitions of similar confluent forms Φ_3 Ψ_1 , Ψ_2 , Ξ_1 , Ξ_2 , of the remaining Appell series F_2 , F_3 and F_4 :

$$(1.2.1) \quad \Phi_1 [a,b;c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} , \quad |x|<1 , \; |y|< \infty ;$$

$$(1.2.2) \quad \Phi_2 [b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} , \quad |x|< \infty , \; |y| < \infty ;$$

In addition to above confluent forms of Appell series all thirteen confluent forms viz. Γ_1, Γ_2, H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11} of the Horn series are also appeared in the literature due to Erdélyi [19]

Kampé de Fériet Series and its Generalization

Just as the Gaussian Series ${}_2F_1$ was generalized to ${}_pF_q$ by increasing the numbers of parameters in denominator and numerator, the four Appell series were unified and generalized by Kampé de Fériet [35], who defined a general hypergeometric series in two variables (see also Appell and Kampé de Fériet [2, p. 150, eq. (29)]).

The notation introduced by Kampé de Fériet [loc. Cit] for his double hypergeometric series of superior order was subsequently abbreviated by Burchnall and Chaundy [6]. For more general double hypergeometric series (than the one defined by Kampé de Fériet) in a slightly modified notation, one may refer to, Srivastava and Panda [97, p. 423, eq. (26)].
A further generalization of the modified Kampé de Fériet series was given by Srivastava and Daoust [94], who indeed defined the extension of the ${}_p\Psi_q$ series of Fox [29] and Wright ([110] and [111]), in two variables.

1.3 <u>Triple Hypergeometric Series</u>. Lauricella [43, p.114] introduced fourteen complete hypergeometric series in three variables of the second order denoted by the symbols F_1, F_2, F_3, ..., F_{14} of which four series F_1, F_2, F_5 and F_9 correspond respectively to the three variables Lauricella series $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ defined by

(1.3.1) $F_A^{(3)}[a,b_1,b_2,b_3;c_1,c_2,c_3;\ x,y,z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}}{(c_1)_m} \frac{(b_1)_m}{(c_2)_n} \frac{(b_2)_n}{(c_3)_n} (b_3)_n \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$$|x| + |y| + |z| < 1 ;$$

(1.3.2) $F_B^{(3)}[a_1,a_2,a_3,b_1,b_2,b_3;c;\ x,y,z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_n (a_3)_p (b_1)_m (b_2)_n (b_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$$|x| < 1, \quad |y| < 1, \quad |z| < 1 ;$$

(1.3.3) $F_C^{(3)}[a,b;c_1,c_2,c_3;x,y,z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b)_{m+n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$$|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} + |z|^{\frac{1}{2}} < 1 ;$$

(1.3.4) $F_D^{(3)}[a,b_1 b_2,b_3;c;x,y,z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b_1)_m (b_2)_n (b_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$$|x| < 1, \quad |y| < 1, \quad |z| < 1 .$$

The set of remaining ten triple series F_3 , F_4 , F_6 , F_7 , F_8 , F_{10} , ... , F_{14} apparently fell into oblivion , except that is an isolated appearance of F_8 in Mayr [49] who came across this triple series while evaluating certain infinite integrals . Saran [85] initiated a systematic study of these ten triple Gaussian series of Lauricella's set and his notations F_E , F_F , ... , F_T defined below now pravail in the literature :

(1.3.5) F_4 or $F_E[a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,y,z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}\,(b_1)_m\,(b_2)_{n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} ,$$

if $|x| < r$, $|y| = s$, $|z| = t$ then the region of convergence is defined as $r + (\sqrt{s} + \sqrt{t}))^2 = 1$.

(1.3.6) F_{14} or $F_F[a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}\,(b_1)_{m+p}\,(b_2)_n}{(c_1)_m\,(c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} ,$$

$$(1 - s)(s - t) = rs ;$$

(1.3.7) F_8 or $F_G[a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;x,y,z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}\,(b_1)_m\,(b_2)_n\,(b_3)_p}{(c_1)_m\,(c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} ;$$

$$r + s = 1 , r + t = 1 ;$$

(1.3.8) $\quad F_3$ or $F_K[a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, y, z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m\,(a_2)_{n+p}\,(b_1)_{m+p}\,(b_2)_n}{(c_1)_m\,(c_2)_n\,(c_3)_p}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!},$$

$$(1-r)(1-s) = t\ ;$$

(1.3.9) $\quad F_{11}$ or $F_M[a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m\,(a_2)_{n+p}\,(b_1)_{m+p}\,(b_2)_n}{(c_1)_m\,(c_2)_{n+p}}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!},$$

$$r + t = 1 = s\ ;$$

(1.3.10) $\quad F_6$ or $F_N[a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m\,(a_2)_n\,(a_3)_p\,(b_1)_{m+p}\,(b_2)_n}{(c_1)_m\,(c_2)_{n+p}}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!},$$

$$(1-r)s + (1-s)t = 0\ ,$$

(1.3.11) $\quad F_{12}$ or $F_P[a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_2; x, y, z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}\,(a_2)_n\,(b_1)_{m+n}\,(b_2)_p}{(c_1)_m\,(c_2)_{n+p}}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!},$$

$$(st - s - t)^2 = 4rst\ ;$$

(1.3.12) $\quad F_{10}$ or $F_R[a_1, a_2, a_1, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}\,(a_2)_n\,(b_1)_{m+p}\,(b_2)_n}{(c_1)_m\,(c_2)_{n+p}}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!},$$

$$s(1-\sqrt{r})^2 + t(1-s) = 0\ ,$$

(1.3.13) F_7 or F_S $[a_1, a_2, a_2, b_1, b_2, b_3; c_1, c_1, c_1; x, y, z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_{n+p} (b_1)_m (b_2)_n (b_3)_p}{(c_1)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$r + s = rs$, $s = t$,

(1.3.14) F_{13} or F_T $[a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; x, y, z]$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_{n+p} (b_1)_{m+p} (b_2)_n}{(c_1)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$r + s = rs + t$.

Srivastava's Triple Series H_A , H_B , H_C . During the further investigation of Lauricella's fourteen hypergeometric series in three variables, Srivastava ([90] , [92]) introduced three new complete hypergeometric series in three variables viz. H_A , H_B and H_C . Here H_C is generalization of Appell's series F_1 , H_B is generalization of Appell's series F_2 while H_A is generalization of both F_1 and F_2 . While transforming Pochhammer's double - loop contour integrals associated with the series F_C and F_F , the following two interesting triple hypergeometric series, viz. G_A and G_B of Horn's type were also introduced by Pandey [78] , where G_A is generalization of Appell's series F_1 and Horn's series G_1 and G_2 ; and G_B is generalization of the Appell series F_1 and the Horn series G_2.

Motivated by above work further in an investigation of the system of partial differntial equations associated with the

triple hypergeometric series H_C , Srivastava [93] introduced a new triple hypergeometric series G_C , which evidently gives the generalization of Appell's series F_1 and Horn's series G_2 and H_1 , Other triple hypergeometric series are introduced in the literature by Dhawan [17] , Samar [84] and Exton ([26], [28]) .

An unification of Lauricalla's fourteen hypergeometric series F_1 , F_2 , ... , F_{14} and the additional series H_A , H_B : H_C was introduced by Srivastava [91] ,

1.4 Quadruple Hypergeometric Series . Until Exton [24] defined and examined a few of their properties, no specific study was made of any hypergeometric functions of four variables apart from the four Lauricella's functions $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$ and $F_D^{(4)}$ and certain of their limiting cases. On account of the large number of such functions which aries from a systematic study of all the possibilities, he restricted himself to those functions which are complete and of the second order and which involve at least one product of the type $(a, k+m+n+p)$ in series representation ; k,m,n,p are indices of quadruple summation . Exton [24] defined 21 quadruple hypergeometric series in the following way :

(1.4.1) $K_1[a,a,a,a,b,b,b,c;d,e_1,e_2,d;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m+n)\ (c,p)}{(d,k+p)\ (e_1,m)\ (e_2,n)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!} ,$$

(1.4.2) $K_2\,[a,a,a,a;\ b,b,b,c;\ d_1,d_2,d_3,d_4;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m+n)\ (c,p)}{(d_1,k)\ (d_2,m)\ (d_3,n)\ (d_4,p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.3) $K_3\ [a,a,a,a;b_1,b_1,b_2,b_2;c_1\,c_2,c_2,c_1;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,k+m)\ (b_2,n+p)}{(c_1,k+p)\ (c_2,m+n)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.4) $K_4\ [a,a,a,a;b_1,b_1,b_2,b_2;c,d_1,d_2,c;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,k+m)\ (b_2,n+p)}{(c,k+p)\ (d_1,m)\ (d_2,n)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.5) $K_5\ [a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,p+m)\ (b_2,n+p)}{(c_1,k)\ (c_2,m)\ (c_3,n)\ (c_4,p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.6) $K_6\ [a,a,a,a;b,b,c_1,c_2;e,d,d,d;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m)\ (c_1,n)\ (c_2,p)}{(e,k)\ (d,m+n+p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.7) $K_7\ [a,a,a,a,b,b,c_1,c_2;d_1,d_2,d_1,d_2;x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m)\ (c_1,n)\ (c_2,p)}{(d_1,k+n)\ (d_2,m+p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.8) $K_8\ [a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m)\ (c_1,n)\ (c_2,p)}{(d,k+n)\ (e_1,m)\ (e_2,p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m+n)\ (c,p)}{(d_1,k)\ (d_2,m)\ (d_3,n)\ (d_4,p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.3) $K_3 [a,a,a,a;b_1,b_1,b_2,b_2;c_1 c_2,c_2,c_1;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,k+m)\ (b_2,n+p)}{(c_1,k+p)\ (c_2,m+n)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.4) $K_4 [a,a,a,a;b_1,b_1,b_2,b_2;c,d_1,d_2,c;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,k+m)\ (b_2,n+p)}{(c,k+p)\ (d_1,m)\ (d_2,n)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.5) $K_5 [a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,p+m)\ (b_2,n+p)}{(c_1,k)\ (c_2,m)\ (c_3,n)\ (c_4,p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.6) $K_6 [a,a,a,a;b,b,c_1,c_2;e,d,d,d;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m)\ (c_1,n)\ (c_2,p)}{(e,k)\ (d,m+n+p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.7) $K_7 [a,a,a,a,b,b,c_1,c_2;d_1,d_2,d_1,d_2;x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m)\ (c_1,n)\ (c_2,p)}{(d_1,k+n)\ (d_2,m+p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.8) $K_8 [a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m)\ (c_1,n)\ (c_2,p)}{(d,k+n)\ (e_1,m)\ (e_2,p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{u^p}{p!},$$

(1.4.9) $K_9 [a,a,a,a,b,b,c_1,c_2;e_1,e_2,d,d;\ x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m)\ (c_1,n)\ (c_2,p)}{(e_1,k)\ (e_2,m)\ (d,n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{u^p}{p!},$$

(1.4.10) $K_{10} [a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_3,d_4;x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b,k+m)\ (c_1,n)\ (c_2,p)}{(d_1,k)\ (d_2,m)\ (d_3,n)\ (d_4,p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{u^p}{p!},$$

(1.4.11) $K_{11} [a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,k)\ (b_2,m)\ (b_3,n)\ (b_4,p)}{(c,k+m+n)\ (d,p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{u^p}{p!},$$

(1.4.12) $K_{12} [a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,k)\ (b_2,m)\ (b_3,n)\ (b_4,p)}{(c_1\ k+m)\ \ (c_2,n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{u^p}{p!},$$

(1.4.13) $K_{13} [a,a,a,a;b_1,b_2,b_3,b_4;c,c,d_1,d_2;x,y,z,u]$

$$= \sum \frac{(a,k+m+n+p)\ (b_1,k)\ (b_2,m)\ (b_3,n)\ (b_4,p)}{(c,k+m)\ \ (d_1,n)\ (d_2,p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{u^p}{p!},$$

(1.4.14) $K_{14} [a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u]$

$$= \sum \frac{(a,k+m+n)\ (c_3,p)\ (b,k+p)\ (c_1,m)\ (c_2,n)}{(d,k+m+n+p\ )} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{u^p}{p!},$$

(1.4.15) $K_{15} [a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u]$

$$= \sum \frac{(a,k+m+n)\,(b_5,p)\,(b_1,k)\,(b_2,m)\,(b_3,n)(b_4,p)}{(c,k+m+n+p)}\,\frac{x^k}{k!}\,\frac{y^m}{m!}\,\frac{z^n}{n!}\,\frac{u^p}{p!},$$

(1.4.16) $K_{16}\,[a_1,a_2,a_3,a_4;b;x,y,z,u]$

$$= \sum \frac{(a_1,k+m)\,(a_2,k+n)\,(a_3,m+p)\,(a_4,n+p)}{(b,k+m+n+p)}\,\frac{x^k}{k!}\,\frac{y^m}{m!}\,\frac{z^n}{n!}\,\frac{u^p}{p!},$$

(1.4.17) $K_{17}\,[a_1,a_2,a_3,b_1,b_2;c;x,y,z,u]$

$$= \sum \frac{(a_1,k+m)\,(a_2,k+n)\,(a_3,m+n)\,(b_1p)\,(b_2,p)}{(c,k+m+n+p)}\,\frac{x^k}{k!}\,\frac{y^m}{m!}\,\frac{z^n}{n!}\,\frac{u^p}{p!},$$

(1.4.18) $K_{18}\,[a_1,a_2,a_3,b_1,b_2;c;x,y,z,u]$

$$= \sum \frac{(a_1,k+m)\,(a_2,k+p)\,(a_3,m+n)\,(b_1,n)\,(b_2,p)}{(c,k+m+n+p)}\,\frac{x^k}{k!}\,\frac{y^m}{m!}\,\frac{z^n}{n!}\,\frac{u^p}{p!},$$

(1.4.19) $K_{19}\,[a_1,a_2,b_1,b_2,b_3,b_4;c;x,y,z,u]$

$$= \sum \frac{(a_1,k+m)\,(a_2,k+n)\,(b_1,m)\,(b_2,n)\,(b_3,p)\,(b_4,p)}{(c,k+m+n+p)}\,\frac{x^k}{k!}\,\frac{y^m}{m!}\,\frac{z^n}{n!}\,\frac{u^p}{p!},$$

(1.4.20) $K_{20}\,[a_1,a_1,b_3,b_4;b_1,b_2,a_2,a_2;c,c,c,c;x,y,z,u]$

$$= \sum \frac{(a_1,k+m)\,(b_3,n)\,(b_4,p)\,(b_1,k)\,(b_2,m)\,(a_2,n+p)}{(c,k+m+n+p)}\,\frac{x^k}{k!}\,\frac{y^m}{m!}\,\frac{z^n}{n!}\,\frac{u^p}{p!},$$

(1,4,21) $K_{21}\,[a,a,b_6,b_5;\ b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u]$

$$= \sum \frac{(a,k+m)\,(b_6,n)\,(b_5,p)\,(b_1,k)\,(b_2,m)\,(b_3,n)\,(b_4,p)}{(c,k+m+n+p)}\,\frac{x^k}{k!}\,\frac{y^m}{m!}\,\frac{z^n}{n!}\,\frac{u^p}{p!}.$$

Recently, Sharma and Parihar [89] further introduced 83 hypergeometric functions of four variables $F_1^{(4)}, \ldots, F_{83}^{(4)}$. It is worthy to note that out of these eighty three functions following ninteen functions had already been introduced by Exton ([24], [27]) in different notations (see also Chandel and Kumar [12]):

$$F_9^{(4)} = K_1 , \quad F_1^{(4)} = K_2 , \quad F_{38}^{(4)} = K_3 , \quad F_{10}^{(4)} = K_4 ,$$

$$F_2^{(4)} = K_5 , \quad F_{59}^{(4)} = K_6 , \quad F_{39}^{(4)} = K_7 , \quad F_{11}^{(4)} = K_8 ,$$

$$F_{12}^{(4)} = K_9 , \quad F_3^{(4)} = K_{10} , \quad F_{60}^{(4)} = K_{11} , \quad F_{40}^{(4)} = K_{12} ,$$

$$F_{13}^{(4)} = K_{13} , \quad F_{77}^{(4)} = K_{14} , \quad F_{78}^{(4)} = K_{15} , \quad F_{79}^{(4)} = K_{16} ,$$

$$F_{82}^{(4)} = K_{19} , \quad F_{81}^{(4)} = K_{20} , \quad F_{83}^{(4)} = K_{21} .$$

Very recently, Chandel, Agrawal and Kumar [13] have also introduced seven more possible hypergeometric functions of four variables $F_{A_1}^{(4)}$, $F_{A_2}^{(4)}$, $F_{A_3}^{(4)}$, $F_{B_1}^{(4)}$, $F_{B_2}^{(4)}$, $F_{C_1}^{(4)}$, $F_{C_2}^{(4)}$.

1.5 Multiple Hypergeometric Functions of Several Variables

Several authors like Green [30], Hermite [31] and Didon [18] have discussed certain specialized hypergeometric

functions. Lauricella [43] approached this topic systematically, and starting with the Appell functions, he proceeded to define and study the following four important functions which bear his name :

(1.5.1) $$F_A^{(n)}[a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a, m_1 + \ldots + m_n)\ (b_1, m_1)\ \ldots\ (b_n, m_n)}{(c_1, m_1)\ \ldots\ (c_n, m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$|x_1| + \ldots + |x_n| < 1 .$$

(1.5.2) $$F_B^{(n)}[a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a_1, m_1)\ \ldots\ (a_n, m_n)\ (b_1, m_1) \ldots (b_n, m_n)}{(c, m_1 + \ldots + m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$|x_1| < 1 , \ \ldots \ , |x_n| < 1 .$$

(1.5.3) $$F_C^{(n)}[a, b; c_1, \ldots, c_n; x_1, \ldots, x_n]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a, m_1 + \ldots + m_n)\ (b, m_1 + \ldots + m_n)}{(c_1, m_1)\ \ldots\ (c_n, m_n)} \frac{x_n^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$|x_1|^{\frac{1}{2}} + \ldots + |x_n|^{\frac{1}{2}} < 1 .$$

(1.5.4) $$F_D^{(n)}[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a, m_1 + \ldots + m_n)\ (b_1, m_1)\ \ldots (b_n, m_n)}{(c, m_1 + \ldots + m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

It is clear that

$$F_A^{(2)} = F_2 \quad , \quad F_B^{(2)} = F_3 \quad , \quad F_C^{(2)} = F_4 \quad , \quad F_D^{(2)} = F_1 \quad ,$$

where F_1 , F_2 , F_3 , F_4 are the Appell series [1, p.296].

The confluent forms of the Lauricella's functions are defined as

(1.5.5) $$\Psi_2^{(n)}[a; c_1, \ldots, c_n; x_1, \ldots, x_n]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a, m_1 + \ldots + m_n)}{(c_1, m_1) \ldots (c_n, m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

(cf. Erdélyi [21, p. 446, eq. (7.2)]) ;

(1.5.6) $$\Phi_2^{(n)}[b_1, \ldots, b_n; c; x_1, \ldots, x_n]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(b_1, m_1) \ldots (b_n, m_n)}{(c, m_1 + \ldots + m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \quad ,$$

(cf. Humbert [34, p. 429] ; also see Appell and Kampé de Fériet [2, p. 134, eq. (34)]) .

It is clear that $\Phi_2^{(2)} = \Phi_2$, $\Psi_2^{(2)} = \Psi_2$,

where Φ_2 and Ψ_2 are confluent hypergeometric series of two variables Erdélyi et al. [19, pp, 225 - 228] .

Further , Srivastava and Exton [96 , p. 373, eq. (12)] introduced confluent series $\Phi_D^{(n)}$.

while Exton [27, p. 43, eq. (2.1.1.4) and (2.1.1.5)] also introduced the confluent series $\Xi_1^{(n)}$ and $\Phi_3^{(n)}$ which are generalizations of Ξ_1 and Φ_3 in two variables.

<u>Generalization of Lauricella Series</u>. An interesting generalization of Lauricella's multiple hypergeometric series $F_A^{(n)}$ and $F_B^{(n)}$ and Horn's double hypergeometric series H_2 was given by Erdelyi [21, p. 13, eq. (28)].

Srivastava and Daoust ([95, p. 494] Also see Srivastava and Manocha [102, p. 64, (18), (19), (20)]) introduced a most generalized multiple hypergeometric series which is defined as

$$(1.5.7) \qquad S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix} [(a):\theta',\dots,\theta^{(n)}]: [(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\Psi',\dots,\Psi^{(n)}]: [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \quad x_1,\dots,x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{\prod_{j=1}^{A}\Gamma(a_j+\sum_{i=1}^{n} m_i\,\theta_j^{(i)}) \prod_{j=1}^{B'}\Gamma(b'_j+m_1\phi'_j)\cdots\prod_{j=1}^{B^{(n)}}\Gamma(b_j^{(n)}+m_n\phi_j^{(n)})}{\prod_{j=1}^{C}\Gamma(c_j+\sum_{i=1}^{n} m_i\,\Psi_j^{(i)}) \prod_{j=1}^{D'}\Gamma(d'_j+m_1\delta'_j)\cdots\prod_{j=1}^{D^{(n)}}\Gamma(d_j^{(n)}+m_n\delta_j^{(n)})}\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

or alternatively by

$$(1.5.8)\quad F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\Psi',\dots,\Psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\dots,x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j, m_1\theta'_j+\dots+m_n\theta^{(n)}_j)\prod_{j=1}^{B'}(b_j, m_1\phi'_j)\dots\prod_{j=1}^{B^{(n)}}(b^{(n)}_j, m_n\phi^{(n)}_j)}{\prod_{j=1}^{C}(c_j, m_1\Psi'_j+\dots+m_n\Psi^{(n)}_j)\prod_{j=1}^{D'}(d'_j, m_1\delta'_j)\dots\prod_{j=1}^{D^{(n)}}(d^{(n)}_j, m_n\delta^{(n)}_j)} \cdot \frac{x_1^{m_1}}{m_1!},\dots,\frac{x_n^{m_n}}{m_n!},$$

where

$$(1.5.9)\quad \theta^{(i)}_j,\ j=1,\dots,A;\ \phi^{(i)}_j,\ j=1,\dots,B^{(i)};\ \Psi^{(i)}_j,\ j=1,\dots,C;$$

$$\delta^{(i)}_j,\ j=1,\dots,D^{(i)};\quad 1\leq i\leq n;$$

and real and positive and (a) is taken to abbreviate the sequence of A parameters $a_1,\dots,a_A$; $(b^{(i)})$ abbreviates the sequence of $B^{(i)}$ parameters $b^{(i)}_1,\dots,b^{(i)}_{B^{(i)}}$, $i=1,\dots,n$; with similar interpretations for (c) and $(d^{(i)})$, $i=1,\dots,n$; etc. For $n=2$ the above series reduces to the series in two variables defined by Srivastava and Daoust [94] For more study one may refer to Srivastava and Karlsson [101].

Other Generalizations of Lauricella's Series . Exton([25], also see [27]) introduced following multiple hypergeometric series related to Lauricella's $F_D^{(n)}$:

(1.5.10) $${}^{(k)}_{(1)}E_D^{(n)}[a,b_1,\ldots,b_n;c,c';x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)\ (b_1,m_1)\ \ldots\ (b_n,m_n)}{(c,m_1+\ldots+m_k)\ (c',m_{k+1}+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} ,$$

(1.5.11) $${}^{(k)}_{(2)}E_D^{(n)}[a,a',b_1,\ldots,b_n;c;x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+..+m_k)\ (a',m_{k+1}+..+m_n)\ (b_1,m_1)..(b_n,m_n)}{(c,m_1+\ \ldots\ +m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

Motivated by Exton's work Chandel [8] introduced following multiple hypergeometric function closely related to Lauricella's $F_C^{(n)}$:

(1.5.12) $${}^{(k)}_{(1)}E_C^{(n)}[a,a',b;c_1,\ldots,c_n;x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)\ (a',m_{k+1}+..+m_n)}{(c_1,m_1)\ \ldots\ (c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

Exton [26 , p. 193 , eq. (1.5)] also introduced the multiple hypergeometric series $D_n^{(p,q)}$ related to Lauricella's $F_D^{(n)}$.

For q = p , it reduces to Exton [23 , p . 86] (Also see Exton

[27 , p. 104, eq. (3.6.1)]) , which is multivariable generalization of the Horn series G_2 .

Exton [27] also considered three other generalizations of the Horn series denoted by ${}^{(p)}H_j^{(n)}$, $j = 2,3,4$. His series ${}^{(p)}H_2^{(n)}$ is simply Erdélyi's series $H_{n,p}$ [21, p. 13, eq. (28)] while for other two remaining generalizations one may refer to Exton [27, p. 97 , (3.5.1) , (3.5.2)] .

Intermediate Lauricella's Functions. By making an appeal to a commendable idea of interpolation between Lauricella's functions Chandel and Gupta [9] introduced the following three multiple hypergeometric functions related to Lauricella's functions :

(1.5.13) $${}^{(k)}F_{AC}^{(n)}[a,b,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n]$$
$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a,m_1+\ldots+m_n)\,(b,m_1+\ldots+m_k)(b_{k+1},m_{k+1})\ldots(b_n,m_n)}{(c_1,m_1)\;\ldots\;(c_n,m_n)}\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

(1.5.14) $${}^{(k)}F_{AD}^{(n)}[a,b_1,\ldots,b_n;c;c_{k+1},\ldots,c_n;x_1,\ldots,x_n]$$
$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a,m_1+\ldots+m_n)\,(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_k)\,(c_{k+1},m_{k+1})\ldots(c_n,m_n)}\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

(1.5.15) $${}^{(k)}F_{BD}^{(n)}[a,a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots m_k)\,(a_{k+1},m_{k+1})\ldots(a_n,m_n)\,(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!} .$$

It is clear that

(1.5.16) $\quad {}^{(0)}F^{(n)}_{AC} = F^{(n)}_{A}, \quad {}^{(1)}F^{(n)}_{AC} = F^{(n)}_{A}, \quad {}^{(n)}F^{(n)}_{AC} = F^{(n)}_{C}$

(1.5.17) $\quad {}^{(0)}F^{(n)}_{AD} = F^{(n)}_{A}, \quad {}^{(1)}F^{(n)}_{AD} = F^{(n)}_{A}, \quad {}^{(n)}F^{(n)}_{AD} = F^{(n)}_{D}$

(1.5.18) $\quad {}^{(0)}F^{(n)}_{BD} = F^{(n)}_{B}, \quad {}^{(1)}F^{(n)}_{BD} = F^{(n)}_{B}, \quad {}^{(n)}F^{(n)}_{BD} = F^{(n)}_{D}$

Chandel and Gupta [9] also introduced the following confluent forms of the above series

(1.5.19) $\quad {}^{(k)}_{(1)}\Phi^{(n)}_{AC}[a,b;c_1,\ldots,c_n;x_1,\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)\,(b,m_1+\ldots+m_k)}{(c_1,m_1)\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}, \quad k \neq n .$$

(1.5.20) $\quad {}^{(k)}_{(2)}\Phi^{(n)}_{AC}[a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)\,(b_{k+1},m_{k+1})\ldots(b_n,m_n)}{(c_1,m_1)\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}, \quad k \neq 0 .$$

(1.5.21) $\quad {}^{(k)}_{(1)}\Phi^{(n)}_{AD}[a,b_1,\ldots,b_n;c;x_1,\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)\,(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_k)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\ , \quad k \neq n\ .$$

(1.5.22) $${}^{(k)}_{(1)}\Phi^{(n)}_{BD}[a,b_1,\ldots,b_n;c;x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)\,(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\ , \quad k \neq n\ .$$

(1.5.23) $${}^{(k)}_{(2)}\Phi^{(n)}_{BD}[a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a_{k+1},m_{k+1})\ldots(a_n,m_n)\,(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\ ,$$

$$k \neq 0\ .$$

Motivated by this work , Karlsson [36] also introduced fourth possible intermediate Lauricella's function defined by

(1.5.24) $${}^{(k)}F^{(n)}_{CD}[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;\ x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)\,(b,m_{k+1}+\ldots+m_n)\,(b_1,m_1)\ldots(b_k,m_k)}{(c,m_1+\ldots+m_k)\,(c_{k+1},m_{k+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\ ,$$

It is clear that

$${}^{(0)}F^{(n)}_{CD} = F^{(n)}_{C} \quad \text{and} \quad {}^{(n)}F^{(n)}_{CD} = F^{(n)}_{D}\ .$$

For a natural further generalization of the(Srivastava-Daoust) generalized Lauricella functions of several complex variables

defined by (1.5.11) or (1.5.12) , one may refer to Srivastava and Panda ([97], p. 271 , eq. (4.1) , [98] , p. 121 , eq. (1.10))

In the present thesis, we introduce and study several confluent hypergeometric functions of multiple hypergeometric functions of several variables (see also Chandel and Vishwakarma [10] and [11]) .

1.6 Applications of Fractional Calculus. The theory and applications of fractional calculus are based largely upon the familiar differential operator ${}_\alpha D_x^\mu$ defined by (cf. e.g. , Oldham and Spanier [77 , p. 49] , Lavoie et al. [44] and Ross [83] ; see also Srivastava and Owa [103 , p. 356])

$$(1.6.1)\quad {}_\alpha D_x^\mu \{f(x)\} = \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_\alpha^x (x-t)^{\mu-1} f(t)\, dt & (\mathrm{Re}(\mu) < 0), \\[2ex] \dfrac{d^m}{dx^m}\, {}_\alpha D_x^{\mu-m} \{f(x)\} & 0 \le \mathrm{Re}(\mu) < m \;; \; m \in N_0 \end{cases}$$

where $N_0 = N \cup \{0\}$ $(N = \{1,2,\ldots\})$.

For $\alpha = 0$, (1.6.1) defines the classical Riemann -Liouville fractional derivative (or integral) of order μ (or $-\mu$) . On the other hand, when $\alpha \to w$, (1.6.1) may be identified with the definition of the familiar Weyl fractional derivative (or integral of order μ (or $-\mu$)) (see, for details, Erdélyi et al. [20 , chapter 13] and Samko et al. [87]) , For the sake of simplicity , the special case of fractional calculus operator ${}_\alpha D_x^\mu$ for $\alpha = 0$, will be written

as D_x^{μ} , i.e. $D_x^{\mu} \equiv {}_0D_x^{\mu}$ $(\mu \in C)$.

The computation of fractional derivatives (and fractional integrals) of special functions of one and more variables is important from the point of view of the usefulness of these results in (for example) the evaluation of series and integrals (cf. ,e.g. , Nishimoto [75] and Srivastava [105] , the derivation of generating functions (Srivastava and Manocha [102 , chapter 5]) and the solution of differential and integral equations (cf. Nishimoto [75], and Srivastava and Buschman [99 , chapter 3] ; see also Mc Bride and Roach [50], Nishimoto [76] , and Srivastava and Saigo [104]) . Motivated by these and avenues of applications, a number of workers have made use of the fractional calculus operator D_x^{μ} in the theory of special functions of one and more variables (see for example [88], [100], [15] etc.) .

In the present thesis, we obtain several fractional derivative formulas involving multiple hypergeometric functions of several variables discussed in this chapter .

1.7 Applications of Multiple Hypergeometric Functions in Statistics

Different distributions have been discussed by various authors, Block and Rao [4] , Carlson [7] , Daley [15], Datt [16] , Kabe [37], Kaufman, Mathai and Saxena [38], Kendall [39] , Khatri and Pillai ([40] , [41]) , Khatri and Srivastava [41] , Littler and Fackerell [45] , Lukacs and Naha [46] , Lukacs [47] , Mathai ([51] to [61]) , Mathai and Rathie ([62] to [66]) , Mathai and Saxena ([67] to [73]) , Miller [74] , Pillai, Al - Ani and Jouris [79] , Pillai and Jouris [80] , Pillai and Nagarsenker [81]

Robbins and Pitman [82] , Strawderman [107], Thaung [108] , and Wilks [109] .

Srivastava and Singhal [106] studied , many of the classical statistical distributions , which were associated with the beta and gamma distributions . Further Exton [27] discussed generalized beta and gamma distributions with other special multi-variate distributions. He also discussed the expectations of some functions involving Lauricella's multiple hypergeometric functions [43] .

In the last part of the thesis , we extend the above work and obtain some probability density functions associated with the multivariate beta and gamma distributions and make their applications to obtain some expectations involving the most generalized multiple hypergeometric function of Srivastava and Daoust [94] defined by (1.5.7) or (1.5.8) (also see Srivastava and Manocha [102] , p. 64) .

Finally, we also derive the moments for these multivariate beta and gamma distributions and discuss their special cases to obtain the results involving other multiple hypergeometric functions of several variables .

1.8. Brief Survey of the Chapters

In the chapter II , we establish relations between hypergeometric functions of three and four variables .

The chapter III deals with the fractional derivatives involving hypergeometric functions of four variables .

In the chapter IV , we derive generating relations for generalized multiple hypergeometric functions of Srivastava and Daoust [95] and discuss their special cases to obtain new generating relations for other multiple hypergeometric functions

In the chapter V , we study Karlsson's multiple hypergeometric function [36] and introduce eleven confluent forms of multiple hypergeometric functions with their applications in obtaing their recurrence relations .

Chapter VI deals with the use of theory of fractional integration to derive Eulerian integral representations for Karlsson's multiple hypergeometric function [36] and for various confluent forms of multiple hypergeometric functions introduced in the chapter V .

In the chapter VII , we evaluate fractional derivatives involving multiple hypergeometric functions of several variables .

In the chapter VIII and IX on Applications of Multiple hypergeometric functions in Statistics , we establish various probability density functions associated with the multivariate beta and gamma distributions and make their applications to obtain variaous expectations involving the multiple hypergeometric functions of several variables. We also derive moments for these multivariate beta and gamma distributions and discuss their special cases .

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SOME RELATIONS BETWEEN HYPERGEOMETRIC FUNCTIONS OF THREE AND FOUR VARIABLES

CHAPTER II

SOME RELATIONS BETWEEN HYPERGEOMETRIC FUNCTIONS OF THREE AND FOUR VARIABLES

2.1 INTRODUCTION : Lauricella [4, p. 114] introduced fourteen complete hypergeometric series in three variables of the second order . He denoted his triple hypergeometric series by the svombles F_1 , F_2 , ... , F_{14} of which F_1 , F_2 , F_5 and F_9 correspond respectively to the three variables Lauricella series $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ respectively.

After a gap of long time, Saran [5] initiated a systemic study of remaining ten series with the notations F_E , F_F , F_G , F_K , F_M , F_N , F_P , F_R , F_S and F_T for F_4 , F_{14} , F_8 , F_3 , F_{11} , F_6 , F_{12} , F_{10} , F_7 , and F_{13} respectively .

Exton [1,2,3] introduced 21 complete hypergeometric series K_1 , K_2 , ... , K_{21} of four variables , Sharma and Parihar [6] introduced 83 complete hypergeometric series F_1 , F_2 ,..., F_{83} of four variables . It is remarkable that out of these 83 series, the following 19 series had already been appeared in the literature due to Exton [1,2,3] in the different notations;

$$F_9^{(4)} = K_1 , \quad F_1^{(4)} = K_2 , \quad F_{38}^{(4)} = K_3 , \quad F_{10}^{(4)} = K_4 , \quad F_2^{(4)} = K_5$$

$$F_{59}^{(4)} = K_6 , \quad F_{39}^{(4)} = K_7 , \quad F_{11}^{(4)} = K_8 , \quad F_{12}^{(4)} = K_9 ,$$

$F_{3}^{(4)} = K_{10}$, $F_{60}^{(4)} = K_{11}$, $F_{40}^{(4)} = K_{12}$, $F_{13}^{(4)} = K_{13}$,

$F_{77}^{(4)} = K_{14}$, $F_{78}^{(4)} = K_{15}$, $F_{79}^{(4)} = K_{16}$, $F_{82}^{(4)} = K_{19}$,

$F_{81}^{(4)} = K_{20}$, $F_{83}^{(4)} = K_{21}$.

In this chapter, we shall establish certain relations involving above hypergeometric functions of three and four variables:

2.2 RELATIONS

Consider

$$(1-u)^{-b_1} F_A^{(3)} (a,b_1,b_2,b_3 ; c_1,c_2,c_3 ; \frac{x}{1-u},y,z)$$

$$= \sum_{q=0}^{\infty} \frac{u^k}{k!} (b_1)_k \; F_A^{(3)}(a,b_1+k,b_2,b_3 ; c_1,c_2,c_3 ;x,y,z)$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p} (b_1)_{m+q} (b_2)_n (b_3)_p (c_4)_q}{(c)_m (c_2)_n (c_3)_p (c_4)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^k}{k!}$$

Therefore

$$(2.21) \quad (1-u)^{-b_1} F_A^{(3)} (a,b_1,b_2,b_3;c_1,c_2,c_3;\frac{x}{1-u},y,z)$$

$$= F_6^{(4)} (a,a,a,c_4,b_1,b_2,b_3,b_1 ; c_1,c_2,c_3,c_4; x,y,z,u),$$

where $|u|<1$, $\left|\frac{x}{1-u}\right| + |y| + |z|<1$.

Applying the same techniques and make slight adjustment in interchanging of variables, we derive the following relationships :

$$(2.2.2)\quad (1-u)^{-a}\quad F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;\frac{x}{1-u},\frac{y}{1-u},\frac{z}{1-u})$$

$$= \quad F_A^{(4)}(a,b_1,b_2,b_3,b_4;c_1,c_2,c_3,b_4\ ;\ x,y,z,u\ )\ ,$$

$$|x| + |y| + |z|+|u|< 1\quad ,\quad |u|< 1\ .$$

$$(2.2.3)\quad (1-u)^{-a_1}\quad F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;\ \frac{x}{1-u},y,z\ )$$

$$= \quad F_{76}^{(4)}(a_1,a_1,a_3,a_2,b_1,b_4,b_3,b_2;c,b_4,c,c;x,u,z,y\ )\ ,$$

$$|u|< 1\quad ,\quad \left|\frac{x}{1-u}\right| < 1\quad ,\quad |y|< 1\quad ,\quad |z|< 1\ .$$

$$(2.2.4)\quad (1-u)^{-a}\quad F_C^{(3)}(a,b\ :\ c_1,c_2,c_3\ ;\ \frac{x}{1-u},\frac{y}{1-u},\frac{z}{1-u})$$

$$= \quad K_2\ (a,a,a,a\ :\ b,b,b,c_4\ :\ c_1,c_2,c_3,c_4\ :x,y,z,u\ )\ ,$$

$$|u|< 1\ ,\ \left|\frac{x}{1-u}\right|^{\frac{1}{2}} + \left|\frac{y}{1-u}\right|^{\frac{1}{2}} + \left|\frac{z}{1-u}\right|^{\frac{1}{2}} < 1\ .$$

$$(2.2.5)\quad (1-u)^{-a}\quad F_D^{(3)}(a,b_1,b_2,b_3\ ;\ c\ ;\ \frac{x}{1-u},\frac{y}{1-u},\frac{z}{1-u}\ )$$

$$= \quad K_{11}\ (a,a,a,a;\ b_1,b_2,b_3,b_4;c,c,c,b_4;\ x,y,z,u\ )\ ,$$

$$|u|< 1\quad ,\quad \left|\frac{x}{1-u}\right| < 1\ ,\ \left|\frac{y}{1-u}\right|< 1\ ,\ \left|\frac{z}{1-u}\right|< 1\ .$$

$$(2.2.6)\quad (1-u)^{-b_1}\quad F_D^{(3)}(a,b_1,b_2,b_3\ ;\ c\ ;\ \frac{x}{1-u},\ y,z\ )$$

$$= \quad F_{64}^{(4)}(a,a,a,c',b_1,b_2,b_3,b_1\ ;\ c,c,c,c';\ x,y,z,u\ )\ ,$$

$$|u|< 1\quad ,\quad \left|\frac{x}{1-u}\right|< 1\quad ,\quad |y|< 1\quad ,\quad |z|< 1\ .$$

(2.2.7) $(1-u)^{-a_1} F_E(a_1,a_1,a_1,b_1,b_2,b_2 ; c_1,c_2,c_3;\frac{x}{1-u},\frac{y}{1-u},\frac{z}{1-u})$

$= K_{10}(a_1,a_1,a_1,a_1; b_2,b_2,b_1,c_4; c_3,c_2,c_1,c_4; z,y,x,u)$,

$|u|<1$, if $|\frac{x}{1-u}|<r$, $|\frac{y}{1-u}|<s$, $|\frac{z}{1-u}|<t$

then $r+(\sqrt{s}+\sqrt{t})^2=1$.

(2.2.8) $(1-u)^{-b_2} F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,\frac{y}{1-u},\frac{z}{1-u})$

$= F_4^{(4)}(a_1,a_1,a_1,c_4,b_2,b_2,b_1,b_2;c_3,c_2,c_1,c_4 ; z,x,y,u)$,

$|u|<1$, if $|x|<r$, $|\frac{y}{1-u}|<s$, $|\frac{z}{1-u}|<t$,

then $r+(\sqrt{s}+\sqrt{t})^2=1$.

(2.2.9) $(1-u)^{-b_1} F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1c_2,c_3; \frac{x}{1-u},y,z)$

$= F_5^{(4)}(a_1,a_1,a_1,c_4,b_2,b_2,b_1,b_1;c_3,c_2,c_1,c_4 ; z,y,x,u)$,

$|u|<1$, if $|\frac{x}{1-u}|<r$, $|y|<s$, $|z|<t$, then $r+(\sqrt{s}+\sqrt{t})^2=1$.

(2.2.10) $(1-u)^{-a_1} F_F(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;\frac{x}{1-u},\frac{y}{1-u},\frac{z}{1-u})$

$= K_8(a_1,a_1,a_1,a_1;b_1,b_1,b_2,c_3 ; c_2,c_1,c_2,c_3;y,x,z,u)$,

$|u|<1$, if $|\frac{x}{1-u}|<r$, $|\frac{y}{1-u}|<s$, $|\frac{z}{1-u}|<t$,

then $(1-s)(s-t)=rs$.

$$(2.2.11) \quad (1-u)^{-b_1} F_F\left(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;\frac{x}{1-u},y,\frac{z}{1-u}\right)$$

$$= F_{15}^{(4)}\left(a_1,a_1,a_1,c_3,b_1,b_1,b_2,b_1;c_2,c_1,c_2,c_3;z,x,y,u\right),$$

$|u|<1$, if $\left|\frac{x}{1-u}\right|<r$, $|y|<s$, $\left|\frac{z}{1-u}\right|<t$,

then $(1-s)(s-t)=rs$.

$$(2.2.12) \quad (1-u)^{-b_2} F_F\left(a,a,a,b_1,b_2,b_1;c_1c_2,c_2;x,\frac{y}{1-u},z\right)$$

$$= F_{17}^{(4)}\left(a,a,a,c_3,b_1,b_1,b_2,b_2;c_2,c_1,c_2,c_3;z,x,y,u\right),$$

$|u|<1$, if $\left|\frac{y}{1-u}\right|<s$, $|x|<r$, $|z|<t$,

then $(1-s)(s-t)=rs$.

$$(2.2.13) \quad (1-u)^{-a} F_G\left(a,a,a,b_1,b_2,b_3;c_1,c_2,c_2;\frac{x}{1-u},\frac{y}{1-u},\frac{z}{1-u}\right)$$

$$= K_{13}\left(a,a,a,a;b_3,b_2,b_1,c_3;c_2,c_2,c_1,c_3;z,y,x,u\right),$$

$|u|<1$, if $\left|\frac{x}{1-u}\right|<r$, $\left|\frac{y}{1-u}\right|<s$, $\left|\frac{z}{1-u}\right|<t$,

then $r+s=1$, $r+t=1$.

$$(2.2.14) \quad (1-u)^{-b_1} F_G\left(a,a,a,b_1,b_2,b_3;c_1,c_2,c_2;\frac{x}{1-u},y,z\right)$$

$$= F_{23}^{(4)}\left(a,a,a,c_3,b_1,b_2,b_3,b_1;c_1,c_2,c_2,c_3;x,y,z,u\right),$$

$|u|<1$, if $\left|\frac{x}{1-u}\right|<r$, $|y|<s$, $|z|<t$,

then $r+s=1$, $r+t=1$.

(2.2.15) $(1-u)^{-b_2} \; F_G\left(a,a,a,b_1,b_2,b_3;c_1,c_2,c_2;x,\frac{y}{1-u},z\right)$

$= \; F^{(4)}_{22}\left(a,a,a,c_3,b_2,b_3,b_1,b_2;c_2,c_2,c_1,c_3;y,z,x,u\right)$,

$|u|<1$, if $|x|<r$, $\left|\frac{y}{1-u}\right|<s$, $|z|<t$,

then $r+s=1$, $r+t=1$.

(2.2.16) $(1-u)^{-a_1} \; F_K\left(a_1, a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;\frac{x}{1-u},y,z\right)$

$= \; F^{(4)}_{8}\left(a, a_1,a_2,a_2,b_1,c_4,b_1,b_2; c_1,c_4,c_3,c_2;x,u,z,y\right)$,

$|u|<1$, if $\left|\frac{x}{1-u}\right|<r$, $|y|<s$, $|z|<t$,

then $(1-r)(1-s)=t$.

(2.2.17) $(1-u)^{-a_2} \; F_K\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,\frac{y}{1-u},\frac{z}{1-u}\right)$

$= \; F^{(4)}_{6}\left(a_2,a_2,a_2,a_1,b_1,b_2,c_4,b_1;c_3,c_2,c_4,c_1; z,y,u,x\right)$,

$|u|<1$, if $|x|<r$, $\left|\frac{y}{1-u}\right|<s$, $\left|\frac{z}{1-u}\right|<t$,

then $(1-s)(1-r)=t$.

(2.2.18) $(1-u)^{-a_1} \; F_M\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;\frac{x}{1-u},y,z\right)$

$= \; F^{(4)}_{31}\left(a_2,a_2,a_1,a_1,b_1,b_2,b_1,c_3;c_2,c_2,c_1,c_3; x,y,z,u\right)$,

$|u|<1$, if $\left|\frac{x}{1-u}\right|<r$, $|y|<s$, $|z|<t$,

then $r + t = 1 - s$.

(2.2.19) $(1-u)^{-a_2} F_M \left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; x, \frac{y}{1-u}, \frac{z}{1-u}\right)$

$= F_{22}^{(4)} (a_2, a_2, a_2, a_1, b_1, b_2, c_3, b_1; c_2, c_2, c_3, c_1 ; z, y, x, u)$,

$|u| < 1$ if $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $\left|\frac{z}{1-u}\right| < t$,

then $r + t = 1 - s$.

(2.2.20) $(1-u)^{-b_1} F_M \left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, y, \frac{z}{1-u}\right)$

$= F_{25}^{(4)} (b_1, b_1, b_1, b_2, a_2, c_3, a_1, a_2; c_2, c_3, c_1, c_2 ; z, u, x, y)$,

$|u| < 1$, if $\left|\frac{x}{1-u}\right| < r$, $|y| < s$, $\left|\frac{z}{1-u}\right| < t$,

then $r + t = 1 - s$.

(2.2.21) $(1-u)^{-a_1} F_N \left(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, y, z\right)$

$= F_{37}^{(4)} (b_1, b_1, c_3, a_2, a_1, a_3, a_1, b_2; c_1, c_2, c_3, c_2 ; x, z, u, y)$,

$|u| < 1$, if $\left|\frac{x}{1-u}\right| < r$, $|y| < s$, $|z| < t$,

then $(1 - r)s + (1 - s)t = 0$,

(2.2.22) $(1-u)^{-a_2} F_N \left(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; x, \frac{y}{1-u}, z\right)$

$= F_{35}^{(4)} (a_2, a_2, b_1, b_1, b_2, c_3, a_3, a_1; c_2, c_3, c_2, c_1 ; y, u, z, x)$,

$|u| < 1$, $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $|z| < t$,

then $(1 - r)s + (1 - s)t = 0$.

(2.2.23) $(1-u)^{-b_1} F_N\left(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;\frac{x}{1-u},y,\frac{z}{1-u}\right)$

$= F^{(4)}_{26}(b_1,b_1,b_1,b_2,a_3,c_3,a_1,a_2\ ;\ c_2,c_3,c_1,c_2;\ z,u,x,y)$,

$|u|<1$, if $\left|\frac{x}{1-u}\right|<r$, $|y|<s$, $\left|\frac{z}{1-u}\right|<t$,

then $(1-r)s+(1-s)t=0$.

(2.2.24) $(1-u)^{-a_1} F_P\left(a_1,a_2,a_1,b_1,b_1,b_2;c_1,c_2,c_2;\frac{x}{1-u},y,\frac{z}{1-u}\right)$

$= F^{(4)}_{24}(a_1,a_1,a_1,a_2,b_1,b_2,c_3,b_1;\ c_1,c_2,c_3,c_2\ ;\ x,z,u,y)$,

$|u|<1$, if $\left|\frac{x}{1-u}\right|<r$, $|y|<s$, $\left|\frac{z}{1-u}\right|<t$,

then $(st-s-t)^2=4rst$.

(2.2.25) $(1-u)^{-a_2} F_P\left(a_1,a_2,a_1,b_1,b_1,b_2;c_1,c_2,c_2;x,\frac{y}{1-u},z\right)$

$= F^{(4)}_{32}(a_2,a_2,a_1,a_1,b_1,c_3,b_1,b_2;\ c_2,c_3,c_1,c_2;\ y,u,x,z)$,

$|u|<1$, if $|x|<r$, $\left|\frac{y}{1-u}\right|<s$, $|z|<t$,

then $(st-s-t)^2=4rst$.

(2.2.26) $(1-u)^{-b_1} F_P\left(a_1,a_2,a_1,b_1,b_1,b_2;c_1,c_2,c_3;\frac{x}{1-u},\frac{y}{1-u},z\right)$

$= F^{(4)}_{24}(b_1,b_1,b_1,b_2,a_1,a_2,c_3,a_1;c_1,c_2,c_3,c_2;\ x,y,u,z)$,

$|u|<1$, if $\left|\frac{x}{1-u}\right|<r$, $\left|\frac{y}{1-u}\right|<s$, $|z|<t$,

then $(st-s-t)^2=4rst$.

$$(2.2.27) \quad (1-u)^{-a_1} \quad F_R(a_1,a_2,a_1,b_1,b_2,b_1,c_1,c_2,c_2;\ \frac{x}{1-u},y,\frac{z}{1-u})$$

$$= \quad F^{(4)}_{20}\ (a_1,a_1,a_1,a_2,b_1,b_1,c_3,b_2;c_1,c_2,c_3,c_2;x,z,u,y) \quad ,$$

$|u|< 1$, if $\left|\frac{x}{1-u}\right|< r$, $\left|\frac{z}{1-u}\right|< t$, $|y|< s$,

then $s(1-\sqrt{r})^2 + t(1-s) = 0$,

$$(2.2.28) \quad (1-u)^{-a_2} \quad F_R\ (a_1,a_2,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,\frac{y}{1-u},z)$$

$$= \quad F^{(4)}_{30}\ (a_1,a_1,a_2,a_2,b_1,b_1,b_2,b_3;c_2,c_1,c_2,c_3\ ;\ z,x,y,u) \quad ,$$

$|u|< 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right|< s$, $|z|< t$,

then $s(1-\sqrt{r})^2 + t(1-s) = 0$.

$$(2.2.29) \quad (1-u)^{-a_1} \quad F_S(a_1,a_2,a_2,b_1,b_2,b_3;c_1;\ \frac{x}{1-u},y,z)$$

$$= \quad F^{(4)}_{72}\ (a_2,a_2,a_1,b_3,b_2,b_1,c_2;\ c_1,c_1,c_1,c_2;\ z,y,x,u) \quad ,$$

$|u|< 1$, if $\left|\frac{x}{1-u}\right|< r$, $|y|< s$, $|z|< t$,

then $r + s = rs$, $s = t$.

$$(2.2.30) \quad (1-u)^{-a_2} \quad F_S\ (a_1,a_2,a_2,b_1,b_2,b_3;c_1,c_1,c_1;\ x,\frac{y}{1-u},\frac{z}{1-u})$$

$$= \quad F^{(4)}_{67}\ (a_2,a_2,a_2,a_1,b_3,b_2,c_2,b_1;\ c_1,c_1,c_2,c_1\ ;\ z,y,u,x) \quad ,$$

$|u|< 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right|< s$, $\left|\frac{z}{1-u}\right|< t$,

then $r + s = rs$, $s = t$.

$$(2.2.31) \quad (1-u)^{-b_2} F_S\left(a_1,a_2,a_2,b_1,b_2,b_3;c_1,c_1,c_1;\ x,\ \frac{y}{1-u},z\right)$$

$$= F_{74}^{(4)}\left(a_2,a_2,c_2,a_1,b_2,b_3,b_2,b_1;c_1,c_1,c_2,c_1;\ y,z,u,x\right),$$

$|u| < 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $|z| < t$,

then $r + s = rs$, $s = t$.

$$(2.2.32) \quad (1-u)^{-a_1} F_T\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_1,c_1:\ \frac{x}{1-u},y,z\right)$$

$$= F_{70}^{(4)}\left(a_2,a_2,a_1,a_1,b_1,b_2,b_1,c_2;c_1,c_1,c_1,c_2;\ z,y,x,u\right),$$

$|u| < 1$, if $\left|\frac{x}{1-u}\right| < r$, $|y| < s$, $|z| < t$,

then $r + s = sr + t$.

$$(2.2.33) \quad (1-u)^{-a_2} F_T\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_1,c_1;x,\ \frac{y}{1-u},\ \frac{z}{1-u}\right)$$

$$= F_{65}^{(4)}\left(a_2,a_2,a_2,a_1,b_1,b_2,c_2,b_1;c_1,c_1,c_2,c_1;\ z,y,u,x\right),$$

$|u| < 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $\left|\frac{z}{1-u}\right| < t$,

then $r + s = rs + t$.

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FRACTIONAL DERIVATIVES OF CERTAIN HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

CHAPTER III

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FRACTIONAL DERIVATIVES OF CERTAIN HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

3.1 Introduction In the previous chapter II , we established some relations between hypergeometric functions of three and four variables .

Recently, Srivastava and Goyal [14] have derived several fractional derivatives formulae involving the multivariable H - function defined by Srivastava and Panda [15, p. 271, eq. (4.1) et seq.] and studied by them see ([16-19]; see also [12]) .

In the present chapter, for special interest, we apply same techniques in order to derive fractional derivatives involving certain hypergeometric functions of four variables $K_1,\ldots,K_{21}$ of Exton [4,5,6] and those functions of Sharma and Parihar [10], which are not included in Exton's functions [4,5] .

3.2 Fractional Derivatives involving One Fractional Derivative Operator

Making an appeal to the formula [9, p. 67]

$$D_x^{\mu}\{x^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, x^{\lambda-\mu} \quad , \quad \mathrm{Re}(\lambda) > -1 \quad . \tag{3.2.1}$$

we have

$$D_t^{\lambda_4-\mu_4}\left\{t^{\lambda_4-1}K_2(a_1,a_1,a_1,\mu_4,b_1,b_1,b_1,b_1;c_1,c_2,c_3,c_4;x,y,z,t)\right\}$$

$$=\sum_{m,n,p,q=0}^{\infty} D_t^{\lambda_4-\mu_4}\left\{t^{\lambda_4-1}\frac{(a_1)_{m+n+p}(\mu_4)_q(b_1)_{m+n+p+q}}{(c_1)_m(c_2)_m(c_3)_p(c_4)_q}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}\cdot\frac{t^q}{q!}\right\}$$

Therefore

$$(3.2.2)\quad D_t^{\lambda_4-\mu_4}\left\{t^{\lambda_4-1}K_2(a_1,a_1,a_1,\mu_4,b_1,b_1,b_1,b_1;c_1,c_2,c_3,c_4;x,y,z,t)\right\}$$

$$=\frac{\Gamma(\lambda_4)}{\Gamma(\mu_4)}t^{\mu_4-1}K_2(a_1,a_1,a_1,\lambda_4,b_1,b_1,b_1,b_1;c_1,c_2,c_3,c_4;x,y,z,t),$$

$Re(\lambda_4)>0$.

Applying the same technique, we derive the following fractional derivatives :

$$(3.2.3)\quad D_t^{\lambda_4-\mu_4}\left\{t^{\lambda_4-1}K_{11}(a_1,a_2,a_3,a_4,b_1,b_1,b_1,b_1;c_1,c_1,c_1,\lambda_4;x,y,z,t)\right\}$$

$$=\frac{\Gamma(\lambda_4)}{\Gamma(\mu_4)}t^{\mu_4-1}K_{11}(a_1,a_2,a_3,a_4,b_1,b_1,b_1,b_1;c_1,c_1,c_1,\mu_4;x,y,z,t),$$

$Re(\lambda_4)>0$.

$$(3.2.4)\quad D_t^{\lambda_4-\mu_4}\left\{t^{\lambda_4-1}K_{15}(a_1,a_1,a_1,\mu_4,b_1,b_2,b_3,b_4;c_1,c_1,c_1,c_1;x,y,z,t)\right\}$$

$$=\frac{\Gamma(\lambda_4)}{\Gamma(\mu_4)}t^{\mu_4-1}K_{15}(a_1,a_1,a_1,\lambda_4,b_1,b_2,b_3,b_4;c_1,c_1,c_1,c_1;x,y,z,t),$$

$Re(\lambda_4)>0$.

(3.2.5) $$D_t^{\lambda_4-\mu_4}\left\{t^{\lambda_4-1}\,F_5^{(4)}(a_1,a_1,a_1,\mu_4,b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,t)\right\}$$

$$=\frac{\Gamma(\lambda_4)}{\Gamma(\mu_4)}\,t^{\mu_4-1}\,F_5^{(4)}(a_1,a_1,a_1,\lambda_4,b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,t),$$

$Re(\lambda_4)>0$.

(3.2.6) $$D_t^{\lambda_4-\mu_4}\left\{t^{\lambda_4-1}\,F_{42}^{(4)}(a_1,a_1,a_1,\mu_4;b_1,b_1,b_2,b_2;c_1,c_2,c_1,c_2;x,y,z,t)\right\}$$

$$=\frac{\Gamma(\lambda_4)}{\Gamma(\mu_4)}\,t^{\mu_4-1}\,F_{42}^{(4)}(a_1,a_1,a_1,\lambda_4;b_1,b_1,b_2,b_2;c_1,c_2,c_1,c_2;x,y,z,t),$$

$Re(\lambda_4)>0$.

(3.2.7) $$D_t^{\lambda_4-\mu_4}\left\{t^{\lambda_4-1}\,F_{68}^{(4)}(a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_2;c_1,c_1,c_1,\lambda_4;x,y,z,t)\right\}$$

$$=\frac{\Gamma(\lambda_4)}{\Gamma(\mu_4)}\,t^{\mu_4-1}\,F_{68}^{(4)}(a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_2;c_1,c_1,c_1,\mu_4;x,y,z,t),$$

$Re(\lambda_4)>0$.

3.3 Use of two Fractional Derivative Operators

In this section, we derive the following relations :

(3.3.1) $$D_y^{\lambda_2-\mu_2}\,D_t^{\lambda_4-\mu_4}\left\{y^{\lambda_4-1}\,t^{\lambda_4-1}\,K_4(a_1,a_1,a_2,a_2,b_1,b_1,b_1,b_1:c_1,\lambda_2,c_1,\lambda_4;x,y,z,t)\right\}$$

$$=\frac{\Gamma(\lambda_2)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_4)}\,y^{\mu_2-1}\,t^{\mu_4-1}\,K_4(a_1,a_1,a_2,a_2,b_1,b_1,b_1,b_1;c_1,\mu_2,c_1,\mu_4;x,y,z,t),$$

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

$$(3.3.2)\quad D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4}\left\{ z^{\lambda_3-1} t^{\lambda_4-1} K_7(a_1,a_1,\mu_3,\mu_4,b_1,b_1,b_1,b_1; c_1,c_2,c_1,c_2;x,y,z,t)\right\}$$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} K_7(a_1,a_1,\lambda_3,\lambda_4,b_1,b_1,b_1,b_1:c_1,c_2,c_1,c_2: x,y,z,t),$$

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.3.3)\quad D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4}\left\{ y^{\lambda_2-1} t^{\lambda_4-1} K_8(a_1,a_1,a_2,a_3,b_1,b_1,b_1,b_1; c_1,\lambda_2,c_1,\lambda_4;x,y,z,t)\right\}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_4)} y^{\mu_2-1} t^{\mu_4-1} K_8(a_1,a_1,a_2,a_3,b_1,b_1,b_1,b_1;c_1,\mu_2,c_1,\mu_4; x,y,z,t),$$

$Re(\lambda_2) > 0$, $Re(\lambda_4) > 0$.

$$(3.3.4)\quad D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4}\left\{ z^{\lambda_3-1} t^{\lambda_4-1} K_{13}(a_1,a_2,a_3,a_4,b_1,b_1,b_1,b_1;c_1,c_1, \lambda_3,\lambda_4; x,y,z,t)\right\}$$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} K_{13}(a_1,a_2,a_3,a_4,b_1,b_1,b_1,b_1;c_1,c_1,\mu_3, \mu_4:x,y,z,t),$$

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.3.5)\quad D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4}\left\{ z^{\lambda_3-1} t^{\lambda_4-1} K_{19}(a_1,a_1,\mu_3,\mu_4,b_1,b_2,b_1,b_3; c_1,c_1,c_1,c_1; x,y,z,t)\right\}$$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} K_{19}(a_1,a_1,\lambda_3,\lambda_4,b_1,b_2,b_1,b_3;c_1,c_1, c_1,c_1; x,y,z,t),$$

$Re(\lambda_4) > 0$, $Re(\lambda_3) > 0$.

(3.3.6) $D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1} K_{21}(a_1,a_1,\mu_3,\mu_4,b_1,b_2,b_3,b_4;\ c_1,c_1,c_1,c_1;x,y,z,t)\}$

$$= \frac{\Gamma(\lambda_3)\ \Gamma(\lambda_4)}{\Gamma(\mu_3)\ \Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} K_{21}(a_1,a_1,\lambda_3,\lambda_4,b_1,b_2,b_3,b_4;c_1,c_1,c_1,c_1;x,y,z,t) ,$$

$\mathrm{Re}(\lambda_3) > 0$, $\mathrm{Re}(\lambda_4) > 0$.

(3.3.7) $D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1} F_4^{(4)} (a_1,a_1,a_1,\mu_4,b_1,b_1,\mu_3,b_1;\ c_1,c_2,c_3,c_4;\ x,y,z,t)\}$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} F_4^{(4)} (a_1,a_1,a_1,\lambda_4,b_1,b_1,\lambda_3,b_1;c_1,c_2,c_3,c_4;x,y,z,t) ,$$

$\mathrm{Re}(\lambda_3) > 0$, $\mathrm{Re}(\lambda_4) > 0$.

(3.3.8) $D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1} F_8^{(4)} (a_1,a_1,a_2,a_2,b_1,\mu_2,b_1,\mu_4;\ c_1,c_2,c_3,c_4;x,y,z,t)\}$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_4)} y^{\mu_2-1} t^{\mu_4-1} F_8^{(4)} (a_1,a_1,a_2,a_2,b_1,\lambda_2,b_1,\lambda_4;c_1,c_2,c_3,c_4;x,y,z,t) ,$$

$\mathrm{Re}(\lambda_2) > 0$, $\mathrm{Re}(\lambda_4) > 0$.

(3.3.9) $D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1} F_{14}^{(4)} (a_1,a_1,a_1,a_2,b_1,b_1,b_1,b_2;\ c_1,\lambda_2,\lambda_3,c_1;\ x,y,z,t)\}$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_4)}\, y^{\mu_2-1}\, t^{\mu_4-1}\, F^{(4)}_{14}\,(a_1,a_1,a_1,a_2,b_1,b_1,b_1,b_2;c_1,\mu_2,\mu_3,c_1;x,y,z,t)\ ,$$

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

$$(3.3.10)\quad D_x^{\lambda_1-\mu_1}\, D_y^{\lambda_2-\mu_2}\left\{ x^{\lambda_1-1}\, y^{\lambda_2-1}\, F^{(4)}_{16}\,(a_1,a_1,a_1,a_2,b_1,b_1,b_2,b_1;\lambda_1,\lambda_2,c_3,c_3;\ x,y,z,t)\right\}$$

$$= \frac{\Gamma(\lambda_1)\,\Gamma(\lambda_2)}{\Gamma(\mu_1)\,\Gamma(\mu_2)}\, x^{\mu_1-1}\, y^{\mu_2-1}\, F^{(4)}_{16}\,(a_1,a_1,a_1,a_2,b_1,b_1,b_2,b_1;\mu_1,\mu_2,c_3,c_3;x,y,z,t)\ ,$$

$Re(\lambda_1)>0$, $Re(\lambda_2)>0$.

$$(3.3.11)\quad D_y^{\lambda_2-\mu_2}\, D_t^{\lambda_4-\mu_4}\left\{ y^{\lambda_2-1}\, t^{\lambda_4-1}\, F^{(4)}_{17}\,(a_1,a_1,a_1,a_2,b_1,b_1,b_2,b_2;c_1,\lambda_2,c_1,\lambda_4;x,y,z,t)\right\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_4)}\, y^{\mu_2-1}\, t^{\mu_4-1}\, F_{17}(a_1,a_1,a_1,a_2,b_1,b_1,b_2,b_2;c_1,\mu_2,c_1,\mu_4;\ x,y,z,t)\ ,$$

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

$$(3.3.12)\quad D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4}\left\{ z^{\lambda_3-1}\, t^{\lambda_4-1}\, F^{(4)}_{21}\,(a_1,a_1,a_1,a_2,b_1,b_1,\mu_3,\mu_4;c_1,c_2,c_3,c_3;x,y,z,t)\right\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_4)}{\Gamma(\mu_3)\,\Gamma(\mu_4)}\, z^{\mu_3-1}\, t^{\mu_4-1}\, F^{(4)}_{21}\,(a_1,a_1,a_1,a_2,b_1,b_1,\lambda_3,\lambda_4;\ c_1,c_2,c_3,c_3;\ x,y,z,t)\ ,$$

$Re(\lambda_3)>0$, $Re(\lambda_4)>0$.

$$(3.3.13)\quad D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1} \cdot F_{27}^{(4)} [a_1,a_1,a_2,a_2,b_1,b_1,b_2,b_2;c_1,\lambda_2,c_1,\lambda_4;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_4)} y^{\mu_2-1} t^{\mu_4-1} F_{27}^{(4)} [a_1,a_1,a_2,a_2,b_1,b_1,b_2,b_2;c_1,\mu_2,c_1,\mu_4;x,y,z,t],$$

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

$$(3.3.14)\quad D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1} \cdot F_{28}^{(4)} [a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_2;c_1,c_1,\lambda_3,\lambda_4;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_3)\,\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} F_{28}^{(4)} [a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_2;c_1,c_1,\mu_3,\mu_4;x,y,z,t],$$

$Re(\lambda_3)>0$, $Re(\lambda_4)>0$.

$$(3.3.15)\quad D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} \{ y^{\lambda_2-1} z^{\lambda_3-1} \cdot F_{29}^{(4)} [a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_2;c_1,\lambda_2,\lambda_3,c_1;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_2)\,(\lambda_3)}{\Gamma(\mu_2)\,(\mu_3)} y^{\mu_2-1} z^{\mu_3-1} F_{29}^{(4)} [a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_2;c_1,\mu_2,\mu_3,c_1;x,y,z,t],$$

$Re(\lambda_2)>0$, $Re(\lambda_3)>0$.

$$(3.3.16)\quad D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1} \cdot F_{30}^{(4)} [a_1,a_1,a_2,a_2,b_1,b_1,\mu_3,\mu_4;c_1,c_2,c_1,c_3;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_3)\,\Gamma(\mu_4)} \cdot z^{\mu_3-1} t^{\mu_4-1} F_{30}^{(4)} [a_1,a_1,a_2,a_2,b_1,b_1,\lambda_3,\lambda_4;c_1,c_2,c_1,c_3;x,y,z,t],$$

$Re(\lambda_3)>0$, $Re(\lambda_4)>0$.

(3.3.17) $D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1}$.

$F_{31}^{(4)}[a_1,a_1,a_2,a_2,b_1,\mu_2,b_1,\mu_4;c_1,c_1,c_2,c_3;x,y,z,t]\}$

$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_4)} \cdot y^{\mu_2-1} t^{\mu_4-1} \cdot F_{31}^{(4)}[a_1,a_1,a_2,a_2,b_1,\lambda_2,b_1,\lambda_4;c_1,c_1,c_2,c_3;x,y,z,t]$,

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

(3.3.18) $D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} \{ y^{\lambda_2-1} z^{\lambda_3-1}$.

$F_{32}^{(4)}[a_1,a_1,a_2,a_2,b_1,b_2,b_2,b_3;c_1,\lambda_2,\lambda_3,c_1;x,y,z,t]\}$

$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_3)}{\Gamma(\mu_2)\Gamma(\mu_3)} y^{\mu_2-1} z^{\mu_3-1} F_{32}^{(4)}[a_1,a_1,a_2,a_2,b_1,b_2,b_2,b_3;c_1,\mu_2,\mu_3,c_1;x,y,z,t]$,

$Re(\lambda_2)>0$, $Re(\lambda_3)>0$.

(3.3.19) $D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1}$.

$F_{34}^{(4)}[a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_3;c_1,\lambda_2,c_1,\lambda_4;x,y,z,t]\}$

$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_4)} y^{\mu_2-1} t^{\mu_4-1} F_{34}^{(4)}[a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_3;c_1,\mu_2,c_1,\mu_4;x,y,z,t]$,

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

(3.3.20) $D_y^{\lambda_2-\mu_2} \cdot D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} \cdot t^{\lambda_4-1}$.

$F_{36}^{(4)}[a_1,a_1,a_2,a_2,b_1,\mu_2,b_1,\mu_4;c_1,c_2,c_3,c_1;x,y,z,t]\}$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_1)}{\Gamma(\mu_2)\cdot\Gamma(\mu_1)} \cdot y^{\mu_2-1}\, t^{\mu_4-1}\, F^{(4)}_{36}[a_1, a_1, a_2, a_2, b_1, \lambda_2, b_1, \lambda_4; c_1, c_2, c_3, c_1; x, y, z, t],$$

$Re(\lambda_2) > 0$, $Re(\lambda_1) > 0$.

$$(3.3.21)\quad D_x^{\lambda_1-\mu_1} D_z^{\lambda_3-\mu_3} \{ x^{\lambda_1-1} \cdot z^{\lambda_3-1} \cdot$$

$$F^{(1)}_{37}[a_1, a_1, a_2, a_3, b_1, b_2, b_1, b_3; \lambda_1, c_2, \lambda_3, c_2; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\,\Gamma(\lambda_3)}{\Gamma(\mu_1)\,\Gamma(\mu_3)} \cdot x^{\mu_1-1}\, z^{\mu_3-1}\, F^{(4)}_{37}[a_1, a_1, a_2, a_3, b_1, b_2, b_1, b_3; \mu_1, c_2, \mu_3, c_2; x, y, z, t],$$

$Re(\lambda_1) > 0$, $Re(\lambda_3) > 0$.

$$(3.3.22)\quad D_z^{\lambda_3-\mu_3} \cdot D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1}\, t^{\lambda_4-1} \cdot$$

$$F^{(4)}_{41}[a_1, a_1, a_1, \mu_4, b_1, b_1, \mu_3, b_1; c_1, c_2, c_1, c_2; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_3)\,\Gamma(\mu_4)} \cdot z^{\mu_3-1}\, t^{\mu_4-1} \cdot F^{(4)}_{41}[a_1, a_1, a_1, \lambda_4, b_1, b_1, \lambda_3, b_1; c_1, c_2, c_1, c_2; x, y, z, t],$$

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.3.23)\quad D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1}\, t^{\lambda_4-1} \cdot$$

$$F^{(4)}_{43}[a_1, a_1, a_1, a_2, b_1, b_1, \mu_3, \mu_4; c_1, c_2, c_1, c_2; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_3)\,\Gamma(\mu_4)}\, z^{\mu_3-1}\, t^{\mu_4-1} \cdot F^{(4)}_{41}[a_1, a_1, a_1, a_2, b_1, b_1, \lambda_3, \lambda_4; c_1, c_2, c_1, c_2; x, y, z, t],$$

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.3.24) $D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1} .$

$$F_{48}^{(4)}[a_1,a_1,a_2,a_2,b_1,b_1,\mu_3,\mu_4;c_1,c_2,c_1,c_2;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} F_{48}^{(4)}[a_1,a_1,a_2,a_2,b_1,b_1,\lambda_3,\lambda_4;c_1,c_2,c_1,c_2;x,y,z,t],$$

$Re(\lambda_3)>0$, $Re(\lambda_4)>0$.

(3.3.25) $D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1} .$

$$F_{49}^{(4)}[a_1,a_1,a_2,a_2,b_1,\mu_2,b_1,\mu_4;c_1,c_1,c_2,c_2;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_4)} \cdot y^{\mu_2-1} t^{\mu_4-1} F_{49}^{(4)}[a_1,a_1,a_2,a_2,b_1,\lambda_2,b_1,\lambda_4;c_1,c_1,c_2,c_2;x,y,z,t],$$

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

(3.3.26) $D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1} .$

$$F_{50}^{(4)}[a_1,a_1,a_2,a_2,b_1,\mu_2,b_1,\mu_4;c_1,c_2,c_1,c_2;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_2)\ (\lambda_4)}{\Gamma(\mu_2)\ (\mu_4)} \cdot y^{\mu_2-1} t^{\mu_4-1} \cdot F_{50}^{(4)}[a_1,a_1,a_2,a_2,b_1,\lambda_2,b_1,\lambda_4;c_1,c_2,c_1,c_2;x,y,z,t],$$

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

(3.3.27) $D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1} .$

$$F_{51}^{(4)}[a_1,a_1,a_2,a_2,b_1,\mu_2,b_1,\mu_4;c_1,c_2,c_2,c_1;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_4)} \cdot y^{\mu_2-1} \; t^{\mu_4-1} \; F^{(4)}_{51}[a_1, a_1, a_2, a_2, b_1, \lambda_2, b_1, \lambda_4; c_1, c_2, c_2, c_1; x, y, z, t],$$

$Re(\lambda_2) > 0$, $Re(\lambda_4) > 0$.

(3.3.28) $D_z^{\lambda_3-\mu_3} \; D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} \; t^{\lambda_4-1}$.

$$F^{(4)}_{53}[a_1, a_1, \mu_3, \mu_4, b_1, b_1, b_2, b_3; c_1, c_2, c_1, c_2; x, y, z, t]\}$$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} \; z^{\mu_3-1} \; t^{\mu_4-1} \; F^{(4)}_{53}[a_1, a_1, \lambda_3, \lambda_4, b_1, b_1, b_2, b_3; c_1, c_2, c_1, c_2; x, y, z, t],$$

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.3.29) $D_y^{\lambda_2-\mu_2} \; D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} \; t^{\lambda_4-1}$.

$$F^{(4)}_{54}[a_1, a_1, a_2, a_3, b_1, \mu_2, b_1, \mu_4; c_1, c_1, c_2, c_2; x, y, z, t]\}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_4)} \; y^{\mu_2-1} \; t^{\mu_4-1} \; F^{(4)}_{54}[a_1, a_1, a_2, a_3, b_1, \lambda_2, b_1, \lambda_4; c_1, c_1, c_2, c_2; x, y, z, t],$$

$Re(\lambda_2) > 0$, $Re(\lambda_4) > 0$.

(3.3.30) $D^{\lambda_3-\mu_3} \; D^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} \; t^{\lambda_4-1}$.

$$F^{(4)}_{55}[a_1, a_1, \mu_3, \mu_4, b_1, b_2, b_1, b_3; c_1, c_2, c_2, c_1; x, y, z, t]\}$$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} \cdot z^{\mu_3-1} \; t^{\mu_4-1} \cdot F^{(4)}_{55}[a_1, a_1, \lambda_3, \lambda_4, b_1, b_2, b_1, b_3; c_1, c_2, c_2, c_1; x, y, z, t],$$

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.3.31)\quad D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1} \cdot$$

$$F_{56}^{(4)} [a_1, a_1, \mu_3, \mu_4, b_1, b_2, b_3, b_4; c_1, c_2, c_1, c_2; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} \cdot F_{56}^{(4)} [a_1, a_1, \lambda_3, \lambda_4, b_1, b_2, b_3, b_4; c_1, c_2, c_1, c_2; x, y, z, t],$$

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.3.32)\quad D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} t^{\lambda_4-1} \cdot$$

$$F_{62}^{(4)} [a_1, a_1, a_1, \mu_4, b_1, b_1, b_2, b_2; c_1, \lambda_2, c_1, c_1; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_4)} y^{\mu_2-1} t^{\mu_4-1} \quad F_{62}^{(4)} [a_1, a_1, a_1, \lambda_4, b_1, b_1, b_2, b_2; c_1, \mu_2, c_1, c_1; x, y, z, t],$$

$Re(\lambda_2) > 0$, $Re(\lambda_4) > 0$.

$$(3.3.33)\quad D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1} \cdot$$

$$F_{63}^{(4)} [a_1, a_1, a_1, a_2, b_1, b_1, \mu_3, \mu_4; c_1, c_2, c_1, c_1; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} \quad F_{63}^{(4)} [a_1, a_1, a_1, a_2, b_1, b_1, \lambda_3, \lambda_4; c_1, c_2, c_1, c_1; x, y, z, t],$$

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.3.34)\quad D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} \{ y^{\lambda_2-1} z^{\lambda_3-1} \cdot$$

$$F_{64}^{(4)} [a_1, a_1, a_1, a_2, b_1, \mu_2, \mu_3, b_1; c_1, c_1, c_1, c_2; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_3)}{\Gamma(\mu_2)\Gamma(\mu_3)} \cdot y^{\mu_2-1} z^{\mu_3-1} \quad F_{64}^{(4)} [a_1, a_1, a_1, a_2, b_1, \lambda_2, \lambda_3, b_1; c_1, c_1, c_1, c_2; x, y, z, t],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$.

(3.3.35) $D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1}$.

$F_{67}^{(4)} [a_1, a_1, a_1, \mu_4, b_1, b_2, b_3, b_4; c_1, c_2, \lambda_3, c_1; x,y,z,t] \}$

$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} \cdot z^{\mu_3-1} t^{\mu_4-1} F_{67}^{(4)} [a_1, a_1, a_1, \lambda_4, b_1, b_2, b_3, b_4; c_1, c_2, \mu_3, c_1; x,y,z,t]$,

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.3.36) $D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ z^{\lambda_3-1} t^{\lambda_4-1}$.

$F_{72}^{(4)} [a_1, a_1, a_2, a_2, b_1, b_1, \mu_3, \mu_4; c_1, c_1, c_1, c_1; x,y,z,t] \}$

$= \frac{\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_3)\Gamma(\mu_4)} z^{\mu_3-1} t^{\mu_4-1} \cdot F_{72}^{(4)} [a_1, a_1, a_2, a_2, b_1, b_1, \lambda_3, \lambda_4; c_1, c_1, c_1, c_1; x,y,z,t]$,

$Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.3.37) $D_z^{\lambda_3-\mu_3} D_y^{\lambda_2-\mu_2} \{ z^{\lambda_3-1} y^{\lambda_2-1}$.

$F_{77}^{(4)} [a_1, a_1, a_1, a_2, b_1, \mu_2, \mu_3, b_1; c_1, c_1, c_1, c_1; x,y,z,t] \}$

$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_3)}{\Gamma(\mu_2)\Gamma(\mu_3)} \cdot z^{\mu_3-1} y^{\mu_2-1} F_{77}^{(4)} [a_1, a_1, a_1, a_2, b_1, \lambda_2, \lambda_3, b_1, c_1, c_1, c_1, c_1; x,y,z,t]$,

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$.

(3.3.38) $D_y^{\lambda_2-1} D_t^{\lambda_4-1} \{ y^{\lambda_2-1} t^{\lambda_4-1}$.

$F_{80}^{(4)} [a_1, a_1, a_2, a_2, b_1, \mu_2, b_1, \mu_4; c_1, c_1, c_1, c_1; x,y,z,t] \}$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_4)}\, y^{\mu_2-1}\, t^{\mu_4-1}\, F^{(4)}_{80}\left[a_1, a_1, a_2, a_2, b_1, \lambda_2, b_1, \lambda_4; c_1, c_1, c_1, c_1; x, y, z, t\right],$$

$Re(\lambda_2)>0$, $Re(\lambda_4)>0$.

3.4. Use of Three Fractional Derivative Operators

Int this section, we derive the following relations :

(3.4.1) $$D_y^{\lambda_2-\mu_2}\, D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4} \left\{ y^{\lambda_2-1}\, z^{\lambda_3-1}\, t^{\lambda_4-1} \cdot K_1\left[a_1, a_1, a_1, \mu_4, b_1, b_1, b_1, b_1; c_1, \lambda_2, \lambda_3, c_1; x, y, z, t\right] \right\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)} \cdot y^{\mu_2-1}\, z^{\mu_3-1}\, t^{\mu_4-1} \cdot K_1\left[a_1, a_1, a_1, \lambda_4, b_1, b_1, b_1, b_1; c_1, \mu_2, \mu_3, c_1; x, y, z, t\right],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.2) $$D_y^{\lambda_2-\mu_2}\, D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4} \left\{ y^{\lambda_2-1}\, z^{\lambda_3-1}\, t^{\lambda_4-1} \cdot K_6\left[a_1, a_1, \mu_3, \mu_4, b_1, b_1, b_1, b_1; c_1, \lambda_2, c_1, c_1; x, y, z, t\right] \right\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)} \cdot y^{\mu_2-1}\, z^{\mu_3-1}\, t^{\mu_4-1} \cdot K_6\left[a_1, a_1, \lambda_3, \lambda_4, b_1, b_1, b_1, b_1; c_1, \mu_2, c_1, c_1; x, y, z, t\right],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.3) $$D^{\lambda_2-\mu_2}\, D^{\lambda_3-\mu_3}\, D^{\lambda_4-\mu_4} \left\{ t^{\lambda_2-1}\, z^{\lambda_3-1}\, t^{\lambda_4-1} \cdot K_{14}\left[a_1, a_1, a_1, \mu_4, b_1, \mu_2, \mu_3, b_1; c_1, c_1, c_1, c_1; x, y, z, t\right] \right\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)}\; y^{\mu_2-1}\; z^{\mu_3-1}\; t^{\mu_4-1}\,.$$

$$K_{14}[a_1, a_1, a_1, \lambda_4, b_1, \lambda_2, \lambda_3, b_1; c_1, c_1, c_1, c_1; x, y, z, t],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.4) $$D_y^{\lambda_2-\mu_2}\, D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1}\; z^{\lambda_3-1}\; t^{\lambda_4-1}\,.$$

$$F_6^{(4)}[a_1, a_1, a_1, \mu_4, b_1, \mu_2, \mu_3, b_1; c_1, c_2, c_3, c_4; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)} \cdot y^{\mu_2-1}\; z^{\mu_3-1}\; t^{\mu_4-1}\,.$$

$$F_6^{(4)}[a_1, a_1, a_1, \lambda_4, b_1, \lambda_2, \lambda_3, b_1, c_1, c_2, c_3, c_4; x, y, z, t],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.5) $$D_y^{\lambda_2-\mu_2}\, D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1}\; z^{\lambda_3-1}\; t^{\lambda_4-1}\,.$$

$$F_{14}^{(4)}[a_1, a_1, a_1, \mu_4, b_1, b_1, b_1, b_2; c_1, \lambda_2, \lambda_3, c_1; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)} \cdot y^{\mu_2-1}\; z^{\mu_3-1}\; t^{\mu_4-1}\,.$$

$$F_{14}^{(4)}[a_1, a_1, a_1, \lambda_4, b_1, b_1, b_1, b_2; c_1, \mu_2, \mu_3, c_1; x, y, z, t],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.6) $$D_y^{\lambda_2-\mu_2}\, D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1}\; z^{\lambda_3-1}\; t^{\lambda_4-1}\,.$$

$$F_{18}^{(4)}[a_1, a_1, a_1, \mu_4, b_1, b_1, b_2, b_2; c_1, \lambda_2, \lambda_3, c_1; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F^{(4)}_{18}[a_1, a_1, a_1, \lambda_4, b_1, b_1, b_2, b_2; c_1, \mu_2, \mu_3, c_1; x, y, z, t] ,$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.7) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} t^{\lambda_4-1}$$

$$F^{(4)}_{19}[a_1, a_1, a_1, \mu_4, b_1, b_1, b_2, b_2; \lambda_1, \lambda_2, c_3, c_3; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_4-1} .$$

$$F^{(4)}_{19}[a_1, a_1, a_1, \lambda_4, b_1, b_1, b_2, b_2; \mu_1, \mu_2, c_3, c_3; x, y, z, t] ,$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_4) > 0$.

(3.4.8) $$D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F^{(4)}_{20}[a_1, a_1, a_1, \mu_4, b_1, b_1, b_2, b_3; c_1, \lambda_2, \lambda_3, c_1; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F^{(4)}_{20}[a_1, a_1, a_1, \lambda_4, b_1, b_1, b_2, b_3; c_1, \mu_2, \mu_3, c_1; x, y, z, t] ,$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.9) $$D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F^{(4)}_{20}[a_1, a_1, a_1, a_2, b_1, b_1, \mu_3, \mu_4; c_1, \lambda_2, c_3, c_1; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)}\; y^{\mu_2-1}\; z^{\mu_3-1}\; t^{\mu_4-1} \,.$$

$$F^{(4)}_{20}\left[a_1, a_1, a_1, a_2, b_1, b_1, \lambda_3, \lambda_4; c_1, \mu_2, c_3, c_1; x, y, z, t\right],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.10) $$D_y^{\lambda_2-\mu_2}\, D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4} \Big\{ y^{\lambda_2-1}\; z^{\lambda_3-1}\; t^{\lambda_4-1} \,.$$

$$F^{(4)}_{44}\left[a_1, a_1, a_1, \mu_4, b_1, \mu_2, \mu_3, b_1; c_1, c_1, c_2, c_2; x, y, z, t\right] \Big\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)}\; y^{\mu_2-1}\; z^{\mu_3-1}\; t^{\mu_4-1} \,.$$

$$F^{(4)}_{44}\left[a_1, a_1, a_1, \lambda_4, b_1, \lambda_2, \lambda_3, b_1; c_1, c_1, c_2, c_2; x, y, z, t\right],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.11) $$D^{\lambda_2-\mu_2}\, D^{\lambda_3-\mu_3}\, D^{\lambda_4-\mu_4} \Big\{ y^{\lambda_2-1}\; z^{\lambda_3-1}\; t^{\lambda_4-1} \,.$$

$$F^{(4)}_{45}\left[a_1, a_1, a_1, \mu_4, b_1, \mu_2, \mu_3, b_1; c_1, c_2, c_2, c_1; x, y, z, t\right] \Big\}$$

$$= \frac{\Gamma(\lambda_2)\,\Gamma(\lambda_3)\,\Gamma(\lambda_4)}{\Gamma(\mu_2)\,\Gamma(\mu_3)\,\Gamma(\mu_4)}\; y^{\mu_2-1}\; z^{\mu_3-1}\; t^{\mu_4-1} \,.$$

$$F^{(4)}_{45}\left[a_1, a_1, a_1, \lambda_4, b_1, \lambda_2, \lambda_3, b_1; c_1, c_2, c_2, c_1; x, y, z, t\right],$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.12) $$D_y^{\lambda_2-\mu_2}\, D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4} \Big\{ y^{\lambda_2-1}\; z^{\lambda_3-1}\; t^{\lambda_4-1}$$

$$F^{(4)}_{61}\left[a_1, a_1, a_1, \mu_4, b_1, b_1, \mu_3, b_1; c_1, \lambda_2, c_1, c_1; x, y, z, t\right] \Big\}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \; y^{\mu_2-1} \; z^{\mu_3-1} \; t^{\mu_4-1} \, .$$

$$F_{61}^{(4)} [\, a_1, a_1, a_1, \lambda_4, b_1, b_1, \lambda_3, b_1; c_1, \mu_2, c_1, c_1; x, y, z, t \,] ,$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.13) $$D_y^{\lambda_2-\mu_2} \, D_z^{\lambda_3-\mu_3} \, D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} \; z^{\lambda_3-1} \; t^{\lambda_4-1} \, .$$

$$F_{69}^{(4)} [\, a_1, a_1, a_2, a_2, b_1, b_1, \mu_3, \mu_4; c_1, \lambda_2, c_1, c_1; x, y, z, t \,] \}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} . \; y^{\mu_2-1} \; z^{\mu_3-1} \; t^{\mu_4-1} \, .$$

$$F_{69}^{(4)} [\, a_1, a_1, a_2, a_2, b_1, b_1, \lambda_3, \lambda_4; c_1, \mu_2, c_1, c_1; x, y, z, t \,] ,$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.4.14) $$D_y^{\lambda_2-\mu_2} \, D_z^{\lambda_3-\mu_3} \, D_t^{\lambda_4-\mu_4} \{ y^{\lambda_2-1} \; z^{\lambda_3-1} \; t^{\lambda_4-1} \, .$$

$$F_{71}^{(4)} [\, a_1, a_1, a_2, a_2, b_1, \mu_2, b_1, \mu_4; c_1, c_1, \lambda_3, c_4; x, y, z, t \,] \}$$

$$= \frac{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \; y^{\mu_2-1} \; z^{\mu_3-1} \; t^{\mu_4-1}$$

$$F_{71}^{(4)} [\, a_1, a_1, a_2, a_2, b_1, \lambda_2, b_1, \lambda_4; c_1, c_1, \mu_3, c_1; x, y, z, t \,] ,$$

$Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

3.5 Use of Four Fractional Derivative Operators

In this section, we derive the following relations:

$$(3.5.1)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$K_2 [a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1 ; \lambda_1, \lambda_2, \lambda_3, \lambda_4 ; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\cdot\Gamma(\lambda_3)\cdot\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$K_2 [a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1 ; \mu_1, \mu_2, \mu_3, \mu_4 ; x, y, z, t] ,$$

$\mathrm{Re}(\lambda_1) > 0$, $\mathrm{Re}(\lambda_2) > 0$, $\mathrm{Re}(\lambda_3) > 0$, $\mathrm{Re}(\lambda_4) > 0$.

$$(3.5.2)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} \cdot D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$K_5 [a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1 ; \lambda_1, \lambda_2, \lambda_3, \lambda_4 ; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$K_5 [a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1 ; \mu_1, \mu_2, \mu_3, \mu_4 ; x, y, z, t] ,$$

$\mathrm{Re}(\lambda_1) > 0$, $\mathrm{Re}(\lambda_2) > 0$, $\mathrm{Re}(\lambda_3) > 0$, $\mathrm{Re}(\lambda_4) > 0$.

$$(3.5.3)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$K_{10} [a_1, a_1, a_2, a_3, b_1, b_1, b_1, b_1 ; \lambda_1, \lambda_2, \lambda_3, \lambda_4 ; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$K_{10} [a_1, a_1, a_2, a_3, b_1, b_1, b_1, b_1 ; \mu_1, \mu_2, \mu_3, \mu_4 ; x, y, z, t] ,$$

$\mathrm{Re}(\lambda_1) > 0$, $\mathrm{Re}(\lambda_2) > 0$, $\mathrm{Re}(\lambda_3) > 0$, $\mathrm{Re}(\lambda_4) > 0$.

$$(3.5.4)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$K_{11} [\mu_1,\mu_2,\mu_3,\mu_4,b_1,b_1,b_1,b_1;c_1,c_1,c_2,c_2;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$K_{12} [\lambda_1,\lambda_2,\lambda_3,\lambda_4,b_1,b_1,b_1,b_1;c_1,c_1,c_2,c_2;x,y,z,t] ,$$

$$Re(\lambda_1)>0 \ , \ Re(\lambda_2)>0 \ , \ Re(\lambda_3)>0 \ , \ Re(\lambda_4)>0 \ .$$

$$(3.5.5)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$K_{13} [\mu_1,\mu_2,\mu_3,\mu_4,b_1,b_1,b_1,b_1;c_1,c_1,c_2,c_3;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$K_{13} [\lambda_1,\lambda_2,\lambda_3,\lambda_4,b_1,b_1,b_1,b_1; c_1,c_1,c_2,c_3;x,y,z,t] ,$$

$$Re(\lambda_1)>0 \ , \ Re(\lambda_2)>0 \ , \ Re(\lambda_3)>0 \ , \ Re(\lambda_4)>0 \ .$$

$$(3.5.6)\quad D^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$K_{15} [a_1,a_1,a_1,a_2,\mu_1,\mu_2,\mu_3,\mu_4;c_1,c_1,c_1,c_1;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$K_{15} [a_1,a_1,a_1,a_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4,c_1,c_1,c_1,c_1;x,y,z,t] ,$$

$$Re(\lambda_1)>0 \ , \ Re(\lambda_2)>0 \ , \ Re(\lambda_3)>0 \ , \ Re(\lambda_4)>0 \ .$$

$$(3.5.7)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$K_{20} [a_1,a_1,a_2,a_2,\mu_1,\mu_2,\mu_3,\mu_4;c_1,c_1,c_1,c_1;x,y,z,t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1}.$$

$$K_{20} [a_1, a_1, a_2, a_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4; c_1, c_1, c_1, c_1; x, y, z, t],$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.8) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1}. K_{21} [a_1, a_1, a_2, a_3, \mu_1, \mu_2, \mu_3, \mu_4; c_1, c_1, c_1, c_1; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} \cdot y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1}.$$

$$K_{21} [a_1, a_1, a_2, a_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4; c_1, c_1, c_1, c_1; x, y, z, t],$$

$Re(\lambda_1) > 0$ $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.9) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1}. F_{55}^{(4)} [a_1, a_1, a_1, a_2, b_1, b_1, b_2, b_2; \lambda_1, \lambda_2, \lambda_3, \lambda_4; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1}.$$

$$F_5^{(4)} [a_1, a_1, a_1, a_2, b_1, b_1, b_2, b_2; \mu_1, \mu_2, \mu_3, \mu_4; x, y, z, t],$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.10) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1}. F_5^{(4)} [a_1, a_1, a_1, a_2, b_1, b_2, b_3, b_1; \lambda_1, \lambda_2, \lambda_3, \lambda_4; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_6^{(4)}[a_1, a_1, a_1, a_2, b_1, b_2, b_3, b_1; \mu_1, \mu_2, \mu_3, \mu_4; x, y, z, t],$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.11) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_7^{(4)} [a_1, a_1, a_2, a_2, b_1, b_2, b_1, b_2; \lambda_1, \lambda_2, \lambda_3, \lambda_4; x, y, z, t]\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_7^{(4)} [a_1, a_1, a_2, a_2, b_1, b_2, b_1, b_2; \mu_1, \mu_2, \mu_3, \mu_4; x, y, z, t],$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.12) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_8^{(4)} [a_1, a_1, a_2, a_2, b_1, b_2, b_1, b_3; \lambda_1, \lambda_2, \lambda_3, \lambda_4; x, y, z, t]\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_8^{(4)} [a_1, a_1, a_2, a_2, b_1, b_2, b_1, b_3; \mu_1, \mu_2, \mu_3, \mu_4; x, y, z, t],$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.13) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_{16}^{(4)} [a_1, a_1, a_1, \mu_4, b_1, b_1, \mu_3, b_1; \lambda_1, \lambda_2, c_3, c_4; x, y, z, t]\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_1)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} \cdot$$

$$F^{(4)}_{16} [a_1, a_1, a_1, \lambda_4, b_1, b_1, \lambda_3, b_1; \mu_1, \mu_2, c_3, c_3; x, y, z, t] ,$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.14) $$D^{\lambda_1-\mu_1} D^{\lambda_2-\mu_2} D^{\lambda_3-\mu_3} D^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} \cdot$$

$$F^{(4)}_{21} [a_1, a_1, a_1, a_2, b_1, b_1, \mu_3, \mu_4; \lambda_1, \lambda_2, c_3, c_3; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} \cdot$$

$$F^{(4)}_{21} [a_1, a_1, a_1, a_2, b_1, b_1, \lambda_3, \lambda_4; \mu_1, \mu_2, c_3, c_3; x, y, z, t] ,$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.15) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} \cdot$$

$$F^{(4)}_{33} [a_1, a_1, a_2, a_2, b_1, \mu_2, b_1, \mu_4; \lambda_1, c_2, \lambda_3, c_2; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} \cdot x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} \cdot$$

$$F^{(4)}_{33} [a_1, a_1, a_2, a_2, b_1, \lambda_2, b_1, \lambda_4; \mu_1, c_2, \mu_3, c_2; x, y, z, t] ,$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.16) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} \cdot$$

$$F^{(4)}_{37} [a_1, a_1, a_2, a_3, b_1, \mu_2, b_1, \mu_4; \lambda_1, c_2, \lambda_3, c_2; x, y, z, t] \}$$

$$\frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)}\, x^{\mu_1-1}\, y^{\mu_2-1}\, z^{\mu_3-1}\, t^{\mu_1-1}\,.$$

$$F^{(1)}_{37}\left[a_1, a_1, a_2, a_3, b_1, \lambda_2, b_1, \lambda_1; \mu_1, c_2, \mu_3, c_2; x, y, z, t\right],$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_1) > 0$.

(3.5.17) $$D_x^{\lambda_1-\mu_1}\, D_y^{\lambda_2-\mu_2}\, D_z^{\lambda_3-\mu_3}\, D_t^{\lambda_4-\mu_4} \left\{ x^{\lambda_1-1}\, y^{\lambda_2-1}\, z^{\lambda_3-1}\, t^{\lambda_4-1}\,. F^{(1)}_{46}\left[a_1, a_1, a_1, a_2, \mu_1, \mu_2, \mu_3, \mu_1; c_1, c_1, c_2, c_2; x, y, z, t\right] \right\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)}\, x^{\mu_1-1}\, y^{\mu_2-1}\, z^{\mu_3-1}\, t^{\mu_4-1}\,.$$

$$F^{(4)}_{46}\left[a_1, a_1, a_1, a_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4; c_1, c_1, c_2, c_2; x, y, z, t\right],$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.18) $$D^{\lambda_1-\mu_1}\, D^{\lambda_2-\mu_2}\, D^{\lambda_3-\mu_3}\, D^{\lambda_4-\mu_4} \left\{ x^{\lambda_1-1}\, y^{\lambda_2-1}\, z^{\lambda_3-1}\, t^{\lambda_4-1}\,. F^{(4)}_{52}\left[a_1, a_1, a_2, a_2, \mu_1, \mu_2, \mu_3, \mu_4; c_1, c_2, c_1, c_2; x, y, z, t\right] \right\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)}\, x^{\mu_1-1}\, y^{\mu_2-1}\, z^{\mu_3-1}\, t^{\mu_4-1}\,.$$

$$F^{(4)}_{52}\left[a_1, a_1, a_2, a_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4; c_1, c_2, c_1, c_2; x, y, z, t\right],$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.5.19)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_{56}^{(4)} [a_1, a_1, a_2, a_3, \mu_1, \mu_2, \mu_3, \mu_4; c_1, c_2, c_1, c_2; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_{56}^{(4)} [a_1, a_1, a_2, a_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4; c_1, c_2, c_1, c_2; x, y, z, t] ,$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.5.20)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_{66}^{(4)} [a_1, a_1, a_1, \mu_4, b_1, \mu_2, \mu_3, b_1; \lambda_1, c_2, c_2, c_2; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_{66}^{(4)} [a_1, a_1, a_1, \lambda_4, b_1, \lambda_2, \lambda_3, b_1; \mu_1 c_2, c_2, c_2; x, y, z, t] ,$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

$$(3.5.21)\quad D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_{67}^{(4)} [a_1, a_1, a_1, a_2, \mu_1, \mu_2, \mu_3, \mu_4; c_1, c_1, c_2, c_1; x, y, z, t] \}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_{67}^{(4)} [a_1, a_1, a_1, a_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4; c_1, c_1, c_2, c_1; x, y, z, t] ,$$

$Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$, $Re(\lambda_4) > 0$.

(3.5.22) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \Big\{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_{75}^{(4)}\left[a_1,a_1,\mu_3,\mu_4,b_1,\mu_2,b_1,b_3;\lambda_1,c_2,c_2,c_2;x,y,z,t\right]\Big\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_{75}^{(4)}\left[a_1,a_1,\lambda_3,\lambda_4,b_1,\lambda_2,b_1,b_3;\mu_1,c_2,c_2,c_2;x,y,z,t\right] ,$$

$Re(\lambda_1)>0$, $Re(\lambda_2)>0$, $Re(\lambda_3)>0$, $Re(\lambda_4)>0$.

(3.5.23) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \Big\{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_{76}^{(4)}\left[a_1,a_1,a_2,a_3,\mu_1,\mu_2,\mu_3,\mu_4;c_1,c_2,c_1,c_1;x,y,z,t\right]\Big\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_{76}^{(4)}\left[a_1,a_1,a_2,a_3,\lambda_1,\lambda_2,\lambda_3,\lambda_4;c_1,c_2,c_1,c_1;x,y,z,t\right],$$

$Re(\lambda_1)>0$ $Re(\lambda_2)>0$ $Re(\lambda_3)>0$ $Re(\lambda_4)>0$.

(3.5.24) $$D_x^{\lambda_1-\mu_1} D_y^{\lambda_2-\mu_2} D_z^{\lambda_3-\mu_3} D_t^{\lambda_4-\mu_4} \Big\{ x^{\lambda_1-1} y^{\lambda_2-1} z^{\lambda_3-1} t^{\lambda_4-1} .$$

$$F_{78}^{(4)}\left[a_1,a_1,a_1,a_2,\mu_1,\mu_2,\mu_3,\mu_4;c_1,c_1,c_1,c_1;x,y,z,t\right]\Big\}$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)} x^{\mu_1-1} y^{\mu_2-1} z^{\mu_3-1} t^{\mu_4-1} .$$

$$F_{78}^{(4)}\left[a_1,a_1,a_1,a_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4;c_1,c_1,c_1,c_1;x,y,z,t\right] ,$$

$Re(\lambda_1)>0$, $Re(\lambda_2)>0$, $Re(\lambda_3)>0$, $Re(\lambda_4)>0$.

R E F E R E N C E S

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GENERATING RELATIONS FOR MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

CHAPTER IV

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GENERATING RELATIONS FOR MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

4.1 INTRODUCTION: Chandel [1] established generating relations for his multiple hypergeometric function ${}^{(k)}E^{(n)}_{(1)\,C}$ related to Lauricella's $F^{(n)}_{C}$ [7] and for Exton's multiple hypergeometric function ${}^{(k)}E^{(n)}_{(1)\,D}$ [4] related to Lauricella's $F^{(n)}_{D}$ [7] Chandel and Gupta [2] introduced three intermediate Lauricella's ${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$ and obtained generating relations involving them .

In this chapter, we obtain generating relations for generalized multiple hypergeometric function of Srivastava and Daoust [8,9] (Also see Srivastava and Karlsson [10,p 37, (21)], Exton [5,p,107])and discuss their special cases to derive new generating relations for ${}^{(k)}E^{(n)}_{(1)\,C}$ of Chandel [1] ,${}^{(k)}E^{(n)}_{(1)\,D}$,${}^{(k)}E^{(n)}_{(2)\,D}$ of Exton [4] ,${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$ and ${}^{(k)}F^{(n)}_{BD}$ of Chandel and Gupta [2] (For ${}^{(k)}E^{(n)}_{(1)\,D}$, ${}^{(k)}E^{(n)}_{(2)\,D}$; ${}^{(k)}E^{(n)}_{(1)\,C}$ also see Exton [5,pp, 89 - 90, (3.4.1) , (3.4.2) , (3.4.3)]) .

4.2. GENERATING RELATIONS

In this section, we derive the following generating relations for generalized multiple hypergeometric function of Srivastava and Daoust [8,9,10] :

$$(4.2.1)\qquad (1-u)^{-a_i}\, F^{A:B^1;\,..;\,B^{(n)}}_{C:D^1;\,..;\,D^{(n)}}\left(\begin{matrix}[(a):\theta^1,\,..,\,\theta^{(n)}]:\\ [(c):\psi^1,\,..,\,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^1):\phi^1];\,..;\,[(b^{(n)}):\phi^{(n)}];\\ [(d^1):\delta^1];\,..;\,[(d^{(n)});\delta^{(n)}];\end{matrix}\ \frac{x_1}{(1-u)^{\theta_i^1}},\ldots,\frac{x_n}{(1-u)^{\theta_i^{(n)}}}\right)$$

$$= \sum_{k=0}^{\infty}\frac{u^k\,(a_i)_k}{k!}\, F^{A:B';\,..;\,B^{(n)}}_{C:D';\,..;\,D^{(n)}}\left(\begin{matrix}[(a);\theta^1,\,..,\,\theta^{(n)}]_i,\\ [(c):\psi',\,..,\,\psi^{(n)}];\end{matrix}\right.$$

$$\left.\begin{matrix}[a_i+k;\,\theta_i^1,\,..,\,\theta_i^{(n)}]:[(b'):\phi'];\,..;[(b^{(n)}):\phi^{(n)}];\\ [(c'):\psi'];\,..;\,[(c^{(n)}):\psi^{(n)}];\end{matrix}\ x_1,\ldots,x_n\right),$$

where $[(a):\theta',\,..,\,\theta^{(n)}]_i$ denotes $[(a):\theta',\,..,\,\theta^{(n)}]$ excluding $[a_i:\theta'_i,\,..,\,\theta_i^{(n)}]$ and $|u|<1$,

$$1+\sum_{j=1}^{C}\psi_j^{(r)}+\sum_{j=1}^{D^{(r)}}\delta_j^{(r)}-\prod_{j=1}^{A}\theta_j^{(r)}-\prod_{j=1}^{B^{(r)}}\phi_j^{(r)}>0\ ,$$

$r=1,\ldots,n,\ i=1,\ldots,A$.

$$(4.2.2)\qquad (1-u)^{-b_j^{(i)}}\; F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]: [(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\dots,\psi^{(n)}]: [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\dots,\frac{x_i}{(1-u)^{\phi_j^{(i)}}},\dots,x_n\right)$$

$$=\sum_{k=0}^{\infty}\frac{u^k}{k!}\, b_j^{(i)}\; F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]: [(b'):\phi'];\dots;[(b^{(i)}):\phi^{(i)}]_j,\ [b_j^{(j)}+k:\phi_j^{(i)}];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\dots,\psi^{(n)}]: [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\dots,x_n\right),$$

where $[b^{(i)}:\phi^{(i)}]_j$ denotes $[(b^{(i)}):\phi^{(i)}]$ excluding $[b_j^{(i)}:\phi_j^{(i)}]$, $j = 1,\dots,B^{(i)}$, and

$|u| < 1$,

$$1+\sum_{r=1}^{C}\psi_r^{(i)}+\sum_{r=1}^{D^{(i)}}\delta_r^{(i)}-\prod_{r=1}^{A}\theta_r^{(i)}-\prod_{r=1}^{B^{(i)}}\phi_r^{(i)}>0,$$

$i = 1,\dots,n$.

4.3 REDUCTION FORMULAE

An appeal to (4.2.1) and (4.2.2) gives following reduction formulae respectively :

$$(4.3.1)\quad (1-u)^{-a_i}\, F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: \\ [(c):\Psi',\ldots,\Psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{x_1}{(1-u)^{\theta_i'}},\ldots,\frac{x_n}{(1-u)^{\theta_i^{(n)}}}\right)$$

$$= F^{A:B';\ldots;B^{(n)},0}_{C:D';\ldots;D^{(n)},0}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)},\Delta^i]: \\ [(c):\Psi',\ldots,\Psi^{(n)},0]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\ -\ ; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\ -\ ;\end{matrix}\ x_1,\ldots,x_n,u\right)$$

where Δ^i stands for Kronecker delta Δ^i_j ; $i,j \in \{1,\ldots,A\}$

$|u| < 1$,

$$1 + \sum_{j=1}^{C}\Psi_j^{(r)} + \sum_{j=1}^{D^{(r)}}\delta_j^{(r)} - \prod_{j=1}^{A}\theta_j^{(r)} - \prod_{j=1}^{B^{(r)}}\Phi_j^{(r)} > 0 ,$$

$r=1,\ldots,n$.

$$(4.3.2) \qquad (1-u)^{-b_j^{(i)}} \; F^{A:B';..;B^{(n)}}_{C:D';..;D^{(n)}} \left(\begin{matrix} [(a):\theta',..,\theta^{(n)}]: [(b'):\Phi'];..;[(b^{(n)}):\Phi^{(n)}]; \\ [(c):\Psi',..,\Psi^{(n)}]: [(d'):\delta'];..;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \; x_1,\ldots,\frac{x_i}{(1-u)^{\Phi_j^{(i)}}},\ldots,x_n \right)$$

$$= F^{A+1:B';..;B^{(n)};0}_{C:D';..;D^{(n)};0} \left(\begin{matrix} [(a):\theta',..,\theta^{(n)},0], [b_j^{(i)}:0,..,\Phi_j^{(i)}, 0,..,1]: [(b'):\Phi'];..;[(b^{(i)}):\Phi^{(i)}]_j;..;[(b^{(n)}):\Phi^{(n)}]; \\ [(c):\Psi',\ldots,\Psi^{(n)},0]: [(d'):\delta'];..;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \; x_1,\ldots,x_n,u \right)$$

where $[(b^{(i)}):\Phi^{(i)}]_j$ denotes $[(b^{(i)}):\Phi^{(i)}]$ excluding pair $[(b_j^{(i)}):\Phi_j^{(i)}]$, $j = 1, \ldots, B^{(i)}$ and $|u|<1$,

$$1 + \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \prod_{j=1}^{A} \theta_j^{(i)} - \prod_{j=1}^{B^{(i)}} \Phi_j^{(i)} > 0$$

$i = 1, \ldots, n$.

4.4 SPECIAL CASES

IN THIS SECTION, specializing the parameters, we obtain the

following those generating relations which have not been traced out yet now for ${}^{(k)}_{(1)}E^{(n)}_{C}$ of Chandel [1] ${}^{(k)}_{(2)}E^{(n)}_{D}$ of Exton [4] and ${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$ of Chandel and Gupta [2] .

(4.4.1) $$(1-t)^{-a}\ {}^{(k)}_{(1)}E^{(n)}_{C}\left[a,a',b;c_1,\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_k}{1-t},x_{k+1},\ldots,x_n\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}(a)_r\ {}^{(k)}_{(1)}E^{(n)}_{C}\left[a+r,a',b;c_1,\ldots,c_n;x_1,\ldots,x_n\right],$$

$|t|<1$, if $\left|\frac{x_i}{1-t}\right|<r_i$, $i=1,\ldots,k$, $|x_j|<r_j$, $j=k+1,\ldots,n$, then $(\sqrt{r_1}+\ldots+\sqrt{r_k})^2+(\sqrt{r_{k+1}}+\ldots+\sqrt{r_n})^2=1$.

(4.4.2) $$(1-t)^{-a'}\ {}^{(k)}_{(1)}E^{(n)}_{C}\left[a,a',b;c_1,\ldots,c_n;x_1,\ldots,x_k,\frac{x_{k+1}}{1-t},\ldots,\frac{x_n}{1-t}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}(b)_r\ {}^{(k)}_{(1)}E^{(n)}_{C}\left[a,a'+r,b;c_1,\ldots,c_n;x_1,\ldots,x_n\right],$$

$|t|<1$, if $|x_i|<r_i$, $i=1,\ldots,k$; $\left|\frac{x_j}{1-t}\right|<r_j$, $j=k+1,\ldots,n$

then $(\sqrt{r_1}+\ldots+\sqrt{r_k})^2+(\sqrt{r_{k+1}}+\ldots+\sqrt{r_n})^2=1$.

(4.4.3) $$(1-t)^{-b_i}\ {}^{(k)}_{(1)}E^{(n)}_{D}\left[a,b_1,\ldots,b_n;c,c';x_1,\ldots,x_{i-1},\frac{x_i}{1-t},x_{i+1},\ldots,x_n\right]$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (b_i)_r \, {}^{(k)}_{(1)}E_D^{(n)} \left[a, b_1, \ldots, b_{i-1}, b_i + r, b_{i+1}, \ldots, b_n; c, c'; x_1, \ldots x_n \right]$$

$|t|<1$, if $|x_j|< r_j$, $j=1,\ldots,n$, but $j \neq i$, $\left|\frac{x_i}{1-t}\right|< r_i$

then $r_1 = \ldots = r_k$, $r_k + r_n = 1$, $i = 1, \ldots, n$

$r_{k+1} = \ldots = r_n$.

(4.4.4) $$(1-t)^{-a} \, {}^{(k)}_{(2)}E_D^{(n)} \left[a, a', b_1, \ldots, b_n; c; \frac{x_1}{1-t}, \ldots, \frac{x_k}{1-t}, x_{k+1}, \ldots, x_n \right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r \, t^r}{r!} \, {}^{(k)}_{(2)}E_D^{(n)} \left[a+r, a', b_1, \ldots, b_n; c; x_1, \ldots, x_n \right] ,$$

$|t|< 1$, if $\left|\frac{x_i}{1-t}\right|< r_i$, $i = 1, \ldots, k$, $|x_j|< r_j$,

$j = k+1, \ldots, n$,

then $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_n$, $r_k \cdot r_n = r_k + r_n$.

(4.4.5) $$(1-t)^{-a'} \, {}^{(k)}_{(2)}E_D^{(n)} \left[a, a', b_1, \ldots, b_n; c; x_1, \ldots, x_k, \frac{x_{k+1}}{1-t}, \ldots, \frac{x_n}{1-t} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(a')_r \, t^r}{r!} \, {}^{(k)}_{(2)}E_D^{(n)} \left[a, a'+r, b_1, \ldots, b_n; c; x_1, \ldots, x_n \right] ,$$

$|t|<1$, if $|x_i| < r_i$, $i = 1, \ldots, k$, $\left|\frac{x_j}{1-t}\right|< r_j$,

$j = k+1, \ldots, n$, then $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_n$,

$r_k \cdot r_n = r_k + r_n$.

$$(4.4.6) \qquad (1-t)^{-b_i}\ {}^{(k)}E^{(n)}_{(2)\,D}\left[a,a',b_1,\ldots,b_n;c;x_1,\ldots,x_{i-1},\frac{x_i}{1-t},x_{i+1},\ldots,x_n\right]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}E^{(n)}_{(2)\,D}\left[a,a',b_1,\ldots,b_i,b_i+r,b_{i+1},\ldots,b_n;x_1,\ldots,x_n\right],$$

$|t|<1$, if $|x_j|<r_j$, $j(\neq i) = 1,\ldots,n$, $\left|\frac{x_i}{1-t}\right|<r_i$,

$i = 1,\ldots,n$, then $r_1 = \cdots = r_k$, $r_{k+1} = \cdots = r_n$,

$r_k \cdot r_n = r_k + r_n$.

$$(4.4.7) \qquad (1-t)^{-b_i}\ {}^{(k)}F^{(n)}_{AC}\left[a,b,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_{k+1},\ldots,x_{i-1},\frac{x_i}{1-t},x_{i+1},\ldots,x_n\right]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}F^{(n)}_{AC}\left[a,b,b_{k+1},\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;x_1,\ldots,x_n\right],$$

$i = k+1,\ldots,n$, $|t|<1$,

$$\left[|x_1|^{\frac12}+\cdots+|x_{i-1}|^{\frac12}+\left|\frac{x_i}{1-t}\right|^{\frac12}+|x_{i+1}|^{\frac12}+\cdots+|x_k|^{1/2}\right]^2 + |x_{k+1}|+\cdots+|x_n|<1 .$$

$$(4.4.8) \qquad (1-t)^{-b_i}\ {}^{(k)}F^{(n)}_{AD}\left[a,b_1,\ldots,b_n;c;c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},\frac{x_i}{1-t},x_{i+1},\ldots,x_n\right]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}F^{(n)}_{AD}\left[a,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right],$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}F_{AD}^{(n)} \left[a, b_1, \ldots, b_{i-1}, b_i+r, b_{i+1}, \ldots, b_n; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n \right],$$

$|t| < 1$, $\max(|x_1| , \ldots, |\frac{x_i}{1-t}|, \ldots, |x_k|) + |x_{k+1}| + \ldots + |x_n| < 1$, if $i = 1, \ldots, k$.

and $\max(|x_1| , \ldots, |x_k|) + |x_{k+1}| + \ldots + |\frac{x_i}{1-t}| + \ldots + |x_n| < 1$

if $i = k+1, \ldots, n$.

$$(4.4.9) \quad (1-t)^{-a_i}\ {}^{(k)}F_{BD}^{(n)} \left[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_k, \frac{x_{k+1}}{1-t}, \ldots, \frac{x_n}{1-t} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(a_i)_r\, t^r}{r!}\ {}^{(k)}F_{BD}^{(n)} \left[a, a_{k+1}, \ldots, a_{i-1}, a_i+r, a_{i+1}, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n \right],$$

$\max(|x_1| , \ldots, |x_k|, |\frac{x_{k+1}}{1-t}| , \ldots, |\frac{x_n}{1-t}|) < 1$,

$i = k+1, \ldots n$, $|t| < 1$.

$$(4.4.10) \quad (1-t)^{-b_i}\ {}^{(k)}F_{BD}^{(n)} \left[a, a_{k+1}, \ldots, a_n; b_1, \ldots, b_n; c; x_1, \ldots, x_{i-1}, \frac{x_i}{1-t}, x_{i+1}, \ldots, x_n \right]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}F_{BD}^{(n)} \left[a, a_{k+1}, \ldots, a_n; b_1, \ldots, b_{i-1}, b_i+r, b_{i+1}, \ldots, b_n; c; x_1, \ldots, x_n \right],$$

$|t| < 1$,

$\max(|x_1| , \ldots, |x_{i-1}| , |\frac{x_i}{1-t}| , |x_{i+1}| , \ldots, |x_n|) < 1$, $i = 1, \ldots, n$.

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KARLSSON'S MULTIPLE HYPERGEOMETRIC FUNCTION AND CONFLUENT FORMS OF CERTAIN GENERALIZED HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

CHAPTER V

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KARLSSON'S MULTIPLE HYPERGEOMETRIC FUNCTION AND CONFLUENT FORMS OF CERTAIN GENERALIZED HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

5.1 **Introduction** Exton [4,6] introduced two multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_{D}$, ${}^{(k)}_{(2)}E^{(n)}_{D}$ related to Lauricella's $F^{(n)}_{D}$ [8] . Prompted by this work Chandel [1] introduced multiple hypergeometric function ${}^{(k)}_{(1)}E^{(n)}_{C}$ related to Lauricella's $F^{(n)}_{C}$ [8] . Further Chandel and Gupta [2] introduced three intermediate Lauricella's functions ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$, ${}^{(k)}F^{(n)}_{AC}$ and derived some of their confluent forms ${}^{(k)}_{(1)}\Phi^{(n)}_{AC}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{AC}$, ${}^{(k)}_{(1)}\Phi^{(n)}_{AD}$, ${}^{(k)}_{(1)}\Phi^{(n)}_{BD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{BD}$. Following Chandel and Gupta [2] , Karlsson [7] introduced one more intermediate Lauricella function ${}^{(k)}F^{(n)}_{CD}$.

A paper from this chapter entitled "Karlsson's multiple hypergeometric function and its confluent forms" has been published in Jñānābha , 19(1989), 173 - 185 .

In the present chapter, we study Karlsson's multiple hypergeometric function ${}^{(k)}F^{(n)}_{CD}$ and introduce some confluent forms of above multiple hypergeometric functions of several variables and derive their generating relations and integral representations with their applications in obtaining their recurrence relations .

5.2 DEFINITIONS OF CONFLUENT FORMS . For confluent forms, we consider

$$(5.2.1)\ \lim_{b_1,\ldots,b_k\to\infty} {}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;\frac{x_1}{b_1}\cdots\frac{x_k}{b_k},x_{k+1},\ldots,x_n\right]$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}\,(b)_{m_{k+1}+\ldots+m_n}}{(c)_{m_1+\ldots+m_k}\,(c_{k+1})_{m_{k+1}}\cdots(c_n)_{m_n}}\cdot\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]\ ,\quad k\neq 0\ .$$

For $k = 0$, it reduces to Lauricella' $F^{(n)}_{C}$.

$$(5.2.2)\ \lim_{b\to\infty} {}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\frac{x_{k+1}}{b}\cdots\frac{x_n}{b}\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\ldots+m_k}(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \cdot \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right],\ k \neq n .$$

For $k = n$, it reduces to Lauricella's $F_D^{(n)}$.

$$(5.2.3)\ \lim_{a\to\infty} {}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;\ \frac{x_1}{a},\ldots,\frac{x_n}{a}\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b)_{m_{k+1}+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\ldots+m_k}(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \cdot \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right],\ k\neq 0\ ,\ k \neq n\ .$$

For $k = n$, it reduces to $\Phi_2^{(n)}$ while for $k = 0$,it reduces to $\Psi_2^{(n)}$.

$$(5.2.4)\ \lim_{c_{k+1},\ldots,c_n\to 0} {}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_k,c_{k+1}x_{k+1},\ldots,x_nc_n\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b)_{m_{k+1}+\ldots+m_n} (b_1)_{m_1} \ldots (b_k)_{m_k}}{(c)_{m_1+\ldots+m_k}} \cdot \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(4)}\Phi^{(n)}_{CD} [a,b,b_1,\ldots,b_k;c;x_1,\ldots,x_n] , \quad k \neq n .$$

For $k = n$, it reduces to Lauricella's $F^{(n)}_D$.

$$(5.2.5) \lim_{c\to\infty} {}^{(k)}F^{(n)}_{CD} [a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;cx_1,\ldots,x_k c,x_{k+1},\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b)_{m_{k+1}+\ldots+m_n} (b_1)_{m_1} \ldots (b_k)_{m_k}}{(c_{k+1})_{m_{k+1}} \ldots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD} [a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n] , \quad k \neq 0 .$$

For $k = 0$, it reduces to $F^{(n)}_C$.

$$(5.2.6) \lim_{b,c\to\infty} {}^{(k)}F^{(n)}_{CD} [a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;cx_1,\ldots cx_k,\frac{x_{k+1}}{b},\ldots,\frac{x_n}{b}]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \cdot \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right], \quad k \neq 0 .$$

For $k = 0$, it reduces to $\Psi_2^{(n)}$.

(5.2.7) $\lim_{c\to\infty} {}^{(k)}F^{(n)}_{AD}\left[a,b_1,\ldots,b_n;c;c_{k+1},\ldots,c_n;cx_1,\ldots,cx_k,x_{k+1},\ldots,x_n\right]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \cdot \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right], \quad k \neq 0 .$$

For $k = 0$, it reduces to $F_A^{(n)}$.

(5.2.8) $\lim_{b_{k+1},\ldots,b_n\to\infty} {}^{(k)}F^{(n)}_{BD}\left[a,a_{k+1},\ldots,a_n;b_1,\ldots,b_n;c;x_1,\ldots,x_k,\frac{x_{k+1}}{b_{k+1}},\ldots,\frac{x_n}{b_n}\right]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_k}(a_{k+1})_{m_{k+1}}\ldots(a_n)_{m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \cdot \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; c; x_1, \ldots, x_n\right], \quad k \neq n .$$

For $k = n$, it reduces to Lauricella's $F_D^{(n)}$ while for $k = 0$, it reduces to $\Phi_2^{(n)}$, For $k = 1$, $n = 2$, it reduces to Φ_2.

$$(5.2.9) \lim_{c' \to \infty} {}^{(k)}_{(1)}E^{(n)}_{D}\left[a, b_1, \ldots, b_n; c, c'; x_1, \ldots, x_k, c' x_{k+1}, \ldots, c' x_n\right]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1+\ldots+m_k}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n\right], \quad k \neq n .$$

For $k = n$, it reduces to Lauricella's $F_D^{(n)}$.

$$(5.2.10) \lim_{a' \to \infty} {}^{(k)}_{(2)}E^{(n)}_{D}\left[a, a', b_1, \ldots, b_n; c; x_1, \ldots, x_k, \frac{x_{k+1}}{a'}, \ldots, \frac{x_n}{a'}\right]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+\ldots+m_k} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \cdot \frac{x_1}{m_1!} \cdots \frac{x_n}{m_n!}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{D} [a, b_1, \ldots, b_n; c; x_1, \ldots, x_n], \quad k \neq n .$$

For $k = n$, it reduces to Lauricella's $F^{(n)}_{D}$, while for $k = 0$, it reduces to $\Phi^{(n)}_{2}$, for $k = 1$, $n = 2$, it reduces to $F_1 [b_1, b_2, a; c; x_1, x_2]$.

$$(5.2.11) \lim_{a' \to \infty} {}^{(k)}_{(1)}E^{(n)}_{C} \left[a, a', b; c_1, \ldots, c_n; x_1, \ldots, x_k, \frac{x_{k+1}}{a'}, \ldots, \frac{x_n}{a'}\right]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_k} (b)_{m_1 + \ldots + m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \cdot \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{C} [a, b; c_1, \ldots, c_n; x_1, \ldots, x_n], \quad k \neq n .$$

For $k = n$, it reduces to $F^{(n)}_{C}$, while for $k = 0$, it reduces to $\Psi^{(n)}_{2}$. For $k = 1$, $n = 2$, it reduces to $\Psi_1 [b, a; c_1, c_2; x_1, x_2]$.

5.3 INTEGRAL REPRESENTATIONS

Making an appeal to Srivastava [10, p. 101]

$$(5.3.1) \qquad (\lambda, m) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-t} \cdot t^{\lambda + m - 1} \, dt$$

and Erdélyi [3, p.13]

$$(5.3.2) \qquad \frac{1}{(\lambda,m)} = \frac{\Gamma(\lambda)}{2\pi i} \int_{-\infty}^{(0+)} e^{t} \, . \, t^{-\lambda-m} \, dt \,,$$

$Re(\lambda) > 0$, $m = 0,1,2,\ldots$,

we derive the following integral representations :

$$(5.3.3) \qquad {}^{(k)}F^{(n)}_{CD} [a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n]$$

$$= \frac{\Gamma(c)}{2\pi i \Gamma(a) \; \Gamma(b)} \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{(0+)} e^{-(s+t-u)} . s^{a-1} \; t^{b-1} \; u^{-c} . \; {}_1F_0[b_1;-;\frac{sx_1}{u}]$$

$$\ldots \; {}_1F_0 [b_k;-;\frac{sx_k}{u}] \; {}_0F_1 [-;c_{k+1};stx_{k+1}] \ldots {}_0F_1[-;c_n;stx_n] \; ds \; dt \; du \,,$$

where $Re(a) > 0$, $Re(b) > 0$.

$$(5.3.4) \qquad {}^{(k)}_{(1)}\Phi^{(n)}_{CD} [a,b;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n]$$

$$= \frac{\Gamma(c)}{2\pi i \; \Gamma(a)\Gamma(b)} \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{(0+)} e^{-(s+t-u)} \; . \; e^{\frac{s}{u}(x_1+\ldots+x_k)} \; s^{a-1} \; t^{b-1} \; u^{-c} .$$

$${}_0F_1 [-;c_{k+1};stx_{k+1}] \ldots {}_0F_1 [-;c_n;stx_n] \; ds \; dt \; du \,,$$

where $Re(a) > 0$, $Re(b) > 0$.

$$(5.3.5)\qquad {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$= \frac{\Gamma(c)}{2\pi i\,\Gamma(a)}\int_0^\infty\int_{-\infty}^{(0+)} e^{-(s-u)}\, s^{a-1}\, u^{-c}\;{}_1F_0\left[b_1;-;\frac{sx_1}{u}\right]\cdots$$

$${}_1F_0\left[b_k;-;\frac{sx_k}{u}\right]\cdot{}_0F_1\left[-;c_{k+1};sx_{k+1}\right]\cdots{}_0F_1\left[-;c_n;sx_n\right]\,ds\,du\,,$$

where $Re(a) > 0$.

$$(5.3.6)\qquad {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$= \frac{\Gamma(c)}{2\pi i\,\Gamma(b)}\int_0^\infty\int_{-\infty}^{(0+)} e^{-(t-u)}\, t^{b-1}\, u^{-c}\;{}_1F_0\left[b_1;-;\frac{x_1}{u}\right]$$

$$\cdots{}_1F_0\left[b_k;-;\frac{x_k}{u}\right]\,{}_0F_1\left[-;c_{k+1};tx_{k+1}\right]\cdots{}_0F_1\left[-;c_n;tx_n\right]\,dt\,du\,,$$

$Re(b) > 0$.

$$(5.3.7)\qquad {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c;\;x_1,\ldots,x_n\right]$$

$$= \frac{\Gamma(c)}{2\pi i\,\Gamma(a)\,\Gamma(b)}\int_0^\infty\int_0^\infty\int_{-\infty}^{(0+)} e^{-(s+t-u)}\, e^{st(x_{k+1}+\cdots+x_n)}\, s^{a-1}\, t^{b-1}\, u^{-c}$$

$${}_1F_0\left[b_1;-;\frac{x_1 s}{u}\right]\cdots\;{}_1F_0\left[b_k;-;\frac{x_k s}{u}\right]\,ds\,dt\,du\,,$$

where $Re(a) > 0$, $Re(b) > 0$.

(5.3.8) $${}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c;\ x_1,\ldots,x_n\right]$$

$$= \frac{1}{\Gamma(a)\ \Gamma(b)\Gamma(b_1)\ldots\Gamma(b_k)} \int_0^\infty \cdots (k+2) \cdots \int_0^\infty e^{-s\left[1-t(x_{k+1}+\ldots+x_n-1)+t_1+\ldots t_k\right]}$$

$$s^{a-1}\ t^{b-1}\ t_1^{b_1-1}\ldots t_k^{b_k-1}\ {}_0F_1\left[-;c;x_1st_1+\ldots+x_kst_k\right]\ ds\ dt\ dt_1\ldots dt_k\ ,$$

$Re(a)>0$, $Re(b)>0$ and $Re(b_i)>0$, $i=1,\ldots,k$.

(5.3.9) $${}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$= \frac{1}{\Gamma(a)\ \Gamma(b)\Gamma(b_1)\ \ldots\Gamma(b_k)} \int_0^\infty \cdots (k+2) \cdots \int_0^\infty e^{-(s+t+t_1+\ldots+t_k)}\ s^{a-1}\ t^{b-1}.$$

$$t_1^{b_1-1}\ \ldots\ t_k^{b_k-1}\ {}_0F_1\left[-;c;st_1x_1+\ldots+st_kx_k\right]\ {}_0F_1\left[-;c_{k+1};stx_{k+1}\right]$$

$$\ldots\ {}_0F_1\left[-;c_n;stx_n\right]\ ds\ dt\ dt_1\ \ldots\ dt_k\ ,$$

$Re(a)>0$, $Re(b)>0$, $Re(b_i)>0$, $i=1,\ldots,k$.

(5.3.10) $${}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b;c,c_{k+1},\ldots,c_n;\ x_1,\ldots,x_n\right]$$

$$= \frac{1}{\Gamma(a)\ \Gamma(b)} \int_0^\infty\int_0^\infty s^{a-1}\ t^{b-1}\ e^{-(s+t)}\ {}_0F_1\left[-;c\ ;sx_1+\ldots+sx_k\right].$$

$${}_0F_1\left[-;c_{k+1};st\ \ x_{k+1}\right]\ldots\ {}_0F_1\left[-;c_n;stx_n\right]\ ds\ dt\ ,$$

$Re(a)>0$, $Re(b)>0$.

(5.3.11) $${}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$= \frac{1}{\Gamma(a)\Gamma(b_1)\dots\Gamma(b_k)} \int_0^\infty \dots(k+1)\dots \int_0^\infty e^{-(s+t_1+\dots+t_k)} s^{a-1} t_1^{b_1-1} \dots t_k^{b_k-1} \cdot$$

$${}_0F_1[-;c;x_1 t_1 s+\dots+x_k t_k s]\ {}_0F_1[-;c_{k+1};x_{k+1}s]\dots {}_0F_1[-;c_n;x_n s] \cdot ds\, dt_1 \dots dt_k ,$$

$\mathrm{Re}(a)>0$, $\mathrm{Re}(b_i)>0$, $i = 1,\dots,k$.

(5.3.12) $${}^{(k)}_{(3)}\Phi^{(n)}_{CD}[b,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n]$$

$$= \frac{1}{\Gamma(b)\Gamma(b_1)\dots\Gamma(b_k)} \int_0^\infty \dots(k+1)\dots \int_0^\infty e^{-(t+t_1+\dots+t_k)} t^{b-1} t_1^{b_1-1} \dots$$

$$t_k^{b_k-1}\ {}_0F_1[-;c;x_1 t_1+\dots+x_k t_k]\ {}_0F_1[-;c_{k+1};x_{k+1}t]\dots {}_0F_1[-;c_n;x_n t] \cdot dt\, dt_1 \dots dt_k ,$$

$\mathrm{Re}(b)>0$, $\mathrm{Re}(b_i)>0$, $i=1,\dots,k$.

(5.3.13) $${}^{(k)}_{(5)}\Phi^{(n)}_{CD}[a,b,b_1,\dots,b_k;c_{k+1},\dots,c_n;x_1,\dots,x_n]$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a-1} t^{b-1} (1-x_1 s)^{-b_1} \dots (1-x_k s)^{-b_k} \cdot$$

$${}_0F_1[-;c_{k+1};x_{k+1}st]\dots {}_0F_1[-;c_n;x_n st]\ ds\, dt ,$$

$\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$.

(5.3.14) $${}^{(k)}_{(5)}\Phi^{(n)}_{CD}[a,b,b_1,\dots,b_k;c_{k+1},\dots,c_n;x_1,\dots,x_n]$$

$$= \frac{1}{\Gamma(a)\Gamma(b)\Gamma(b_1)\dots\Gamma(b_k)} \int_0^\infty \dots(k+2)\dots \int_0^\infty e^{-[s+t+t_1(1-sx_1)+\dots+t_k(1-sx_k)]} \cdot$$

$$s^{a-1} t^{b-1} t_1^{b_1-1} \dots t_k^{b_k-1}\ {}_0F_1[-;c_{k+1};x_{k+1}st]\dots {}_0F_1[-;c_n;x_n st] \cdot ds\, dt\, dt_1 \dots dt_k ,$$

$Re(a) > 0$, $Re(b) > 0$, $Re(b_j) > 0$, $j = 1,\ldots,k$.

(5.3.15) $$ {}^{(k)}_{(5)}\Phi^{(n)}_{CD} [a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n] $$

$$ = \frac{1}{\Gamma(a)} \int_0^\infty e^{-s} s^{a-1} (1-x_1 s)^{-b_1} \ldots (1-x_k s)^{-b_k} {}_0F_1[-;c_{k+1};sx_{k+1}] \ldots {}_0F_1[-;c_n;sx_n] \, ds , $$

$Re(a) > 0$.

(5.3.16) $$ {}^{(k)}_{(6)}\Phi^{(n)}_{CD} [a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n] $$

$$ = \frac{1}{\Gamma(a)\Gamma(b_1)\ldots\Gamma(b_k)} \int_0^\infty \ldots (k+1) \ldots \int_0^\infty s^{a-1} t_1^{b_1-1} \ldots t_k^{b_k-1} $$

$$ e^{-[s+t_1(1-sx_1)+\ldots+t_k(1-sx_k)]} \cdot {}_0F_1[-;c_{k+1};x_{k+1}s] \ldots {}_0F_1[-;c_n x_n s] \cdot ds\, dt_1 \ldots dt_k , $$

$Re(a) > 0$, $Re(b_j) > 0$, $j = 1,\ldots,k$.

(5.3.17) $$ {}^{(k)}_{(2)}\Phi^{(n)}_{AD} [a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_n] $$

$$ = \frac{1}{\Gamma(a)} \int_0^\infty e^{-s} s^{a-1} \prod_{j=1}^{k} (1-x_j s)^{-b_j} {}_1F_1[b_{k+1};c_{k+1};x_{k+1}s] \ldots {}_1F_1[b_n;c_n;x_n s] \, ds , $$

$Re(a) > 0$.

(5.3.18) $$ {}^{(k)}_{(2)}\Phi^{(n)}_{AD} [a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_n] $$

$$= \frac{1}{\Gamma(a)\Gamma(b_1)\ldots\Gamma(b_n)} \int_0^\infty \ldots (n+1) \ldots \int_0^\infty e^{-[s+t_1(1-x_1 s)+\ldots+t_k(1-s x_k)+t_{k+1}+\ldots+t_n]} .$$

$$s^{a-1}\, t_1^{b_1-1} \ldots t_n^{b_n-1}\; {}_0F_1[-;c_{k+1};x_{k+1}t_{k+1}s] \ldots {}_0F_1[-;c_n;x_n t_n s]\; ds\cdot dt_1 \ldots dt_n ,$$

$Re(a) > 0$, $Re(b_j) > 0$, $j = 1, \ldots, n$.

(5.3.19) $${}^{(k)}_{(3)}\Phi^{(n)}_{BD}[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; c; x_1, \ldots, x_n]$$

$$= \frac{\Gamma(c)}{2\pi i \Gamma(a)} \int_0^\infty \int_{-\infty}^{(0+)} e^{-(s-u)}\, s^{a-1}\, u^{-c} \prod_{i=1}^{k} \left(1 - \frac{s x_i}{u}\right)^{-b_i} \prod_{j=k+1}^{n} \left(1 - \frac{x_j}{u}\right)^{-a_j} .$$

$$ds\, du ,$$

$Re(a) > 0$.

(5.3.20) $${}^{(k)}_{(3)}\Phi^{(n)}_{BD}[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; c; x_1, \ldots, x_n]$$

$$= \frac{1}{\Gamma(a)\Gamma(b_1)\ldots\Gamma(b_k)\Gamma(a_{k+1})\ldots\Gamma(a_n)} \int_0^\infty \ldots (n+1) \ldots \int_0^\infty e^{-[s+t_1+\ldots+t_k+s_{k+1}+\ldots+s_n]} .$$

$$s^{a-1} \prod_{i=1}^{k} (t_i)^{b_i-1} \prod_{j=k+1}^{n} (s_j)^{a_j-1} \cdot {}_0F_1[-;c;x_1 s t_1+\ldots+x_k s t_k+s_{k+1}x_{k+1}+\ldots+x_n s_n]$$

$$\cdot ds\, dt_1 \ldots dt_k \cdot ds_{k+1} \ldots ds_n ,$$

$Re(a) > 0$, $Re(b_i) > 0$, $i = 1, \ldots, k$, $Re(a_j) > 0$, $j = k+1, \ldots, n$.

(5.3.21) $${}^{(k)}_{(1)}\Phi^{(n)}_{D}[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n]$$

$$= \frac{\Gamma(c)}{2\pi i \Gamma(a)} \int_0^\infty \int_{-\infty}^{(0+)} s^{a-1}\, u^{-c}\, e^{-(s-u)} \prod_{i=1}^{k} \left(1 - \frac{s x_i}{u}\right)^{-b_i} \prod_{j=k+1}^{n} (1 - x_j s)^{-b_j}\, ds\, du ,$$

$Re(a) > 0$, $Re(c) > 0$.

(5.3.22) $${}^{(k)}_{(1)}\Phi_D^{(n)}\left[a,b_1,\ldots,b_n;c;\ x_1,\ldots,x_n\right]$$

$$= \frac{1}{\Gamma(a)\Gamma(b_1)\ldots\Gamma(b_n)}\int_0^\infty\ldots(n+1)\ldots\int_0^\infty s^{a-1}\ t_1^{b_1-1}\ldots t_n^{b_n-1}\,.$$

$$e^{-\left[t_1+\ldots+t_n+s(1-t_{k+1}x_{k+1}-\ldots-t_nx_n)\right]}\ {}_0F_1\left[-;c;x_1t_1s+\ldots+x_kt_ks\right]\,.$$

$$ds\ dt_1\ldots dt_n\ ,$$

$Re(a) > 0$, $Re(b_i) > 0$, $i=1,\ldots,n$.

(5.3.23) $${}^{(k)}_{(1)}\Phi_C^{(n)}\left[a,b;c_1,\ldots,c_n;x_1,\ldots,x_n\right]$$

$$= \frac{1}{\Gamma(a)\ \Gamma(b)}\int_0^\infty\int_0^\infty e^{-(s+t)}\ s^{a-1}\ t^{b-1}\ {}_0F_1\left[-;c_1;stx_1\right]\ldots{}_0F_1\left[-;c_k;stx_k\right].$$

$${}_0F_1\left[-;c_{k+1};tx_{k+1}\right]\ldots{}_0F_1\left[-;c_n;tx_n\right]\ ds\ dt\ ,$$

$Re(a) > 0$, $Re(b) > 0$.

(5.3.24) $${}^{(k)}_{(2)}\Phi_D^{(n)}\left[a,b_1,\ldots,b_n;c;x_1,\ldots,x_n\right]$$

$$= \frac{\Gamma(c)}{2\pi i\,\Gamma(a)}\int_0^\infty\int_{-\infty}^{(0+)} e^{-(s-u)}\ s^{a-1}\ u^{-c}\prod_{i=1}^{k}\left(1-\frac{sx_i}{u}\right)^{-b_i}\prod_{j=k+1}^{n}\left(1-\frac{x_j}{u}\right)^{-b_j}.$$

$$ds\ du\ ,$$

$Re(a) > 0$, $Re(c) > 0$.

(5.3.25) $${}^{(k)}_{(2)}\Phi_D^{(n)}\left[a,b_1,\ldots,b_n;c;x_1,\ldots,x_n\right]$$

$$= \frac{1}{\Gamma(a)\Gamma(b_1)\ldots\Gamma(b_n)}\int_0^\infty\ldots(n+1)\ldots\int_0^\infty e^{-(s+t_1+\ldots+t_n)}\ s^{a-1}\ t_1^{b_1-1}\ldots t_n^{b_n-1}\,.$$

$${}_0F_1\left[-;c;st_1x_1+\ldots+st_kx_k+t_{k+1}x_{k+1}+\ldots+t_nx_n\right]\ ds.dt_1\ldots dt_n\ ,$$

$Re(a) > 0$, $Re(b_j) > 0$, $j=1,\ldots,n$.

5.4 RECURRENCE RELATIONS

Making an appeal to Exton [5,p.115(3.5)]

$$(5.4.1) \quad {}_0F_1[-;c-1;x] - {}_0F_1[-;c;x] - \frac{x}{c(c-1)}\ {}_0F_1[-;c+1;x] = 0$$

and Slater [9,p.19]

$$(5.4.2) \quad c\ {}_1F_1[a;c;x] - c\ {}_1F_1[a-1;c;x] - x\ {}_1F_1[a;c+1;x] = 0$$

$$(5.4.3) \quad (1+a-c)\ {}_1F_1[a;c;x] - a\ {}_1F_1[a+1;c;x] + (c-1)\ {}_1F_1[a;c-1;x] = 0,$$

we derive following recurrence relations from results of section 5.3 :

$$(5.4.4) \quad {}^{(k)}F^{(n)}_{CD}[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n]$$

$$= {}^{(k)}F^{(n)}_{CD}[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_{k+j-1},c_{k+j}-1,c_{k+j+1},\ldots,c_n;x_1,\ldots,x_n]$$

$$- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)}\ {}^{(k)}F^{(n)}_{CD}[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_{k+j-1},c_{k+j}+1,$$

$$c_{k+j+1},\ldots,c_n;x_1,\ldots,x_n] ,$$

where $j = 1,\ldots,n-k$.

$$(5.4.5) \quad {}^{(k)}_{(1)}\Phi^{(n)}_{CD}[a,b;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n]$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b;c,c_{k+1},\dots,c_{k+j-1},c_{k+j}-1,c_{k+j+1},\dots,c_n;x_1,\dots,x_n\right]$$

$$-\frac{x_{j+k}}{c_{k+j}(c_{k+j}-1)}\;{}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b;c,c_{k+1},\dots,c_{k+j-1},c_{k+j}+1,c_{k+j+1},\dots,c_n;x_1,\dots x_n\right],$$

where $j = 1,\dots, n-k$.

(5.4.6) $${}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n\right]$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\dots,b_k;c,c_{k+1},\dots,c_{k+j-1},c_{k+j}-1,c_{k+j+1},\dots,c_n;x_1,\dots,x_n\right]$$

$$-\frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)}\;{}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\dots,b_k;c,c_{k+1},\dots,c_{k+j-1},c_{k+j}+1,c_{k+j+1},\dots,c_n;x_1,\dots,x_n\right],$$

where $j = 1,\dots, n-k$.

(5.4.7) $${}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n\right]$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b,b_1,\dots,b_k;c,c_{k+1},\dots,c_{k+j-1},c_{k+j}-1,c_{k+j+1},\dots,c_n;x_1,\dots,x_n\right]$$

$$-\frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)}\;{}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b,b_1,\dots,b_k;c,c_{k+1},\dots,c_{k+j-1},c_{k+j}+1,c_{k+j+1},\dots,c_n;x_1,\dots,x_n\right],$$

$j = k+1,\dots,n-k$.

(5.4.8) $${}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,b_1,\dots,b_k;c_{k+1},\dots,c_n;\ x_1,\dots,x_n\right]$$

$$= {}^{(k)}_{(5)}\phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_{k+j-1},c_{j+k}-1,c_{k+j+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)}\ {}^{(k)}_{(5)}\phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_{k+j-1},c_{k+j}+1,\right.$$
$$\left. c_{k+j+1},\ldots,c_n;x_1,\ldots,x_n\right] ,$$

where $j = 1,\ldots,n-k$.

(5.4.9) $${}^{(k)}_{(6)}\phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$= {}^{(k)}_{(6)}\phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c_{k+1},\ldots,c_{k+j+1},c_{k+j}-1,c_{k+j+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)}\ {}^{(k)}_{(6)}\phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c_{k+1},\ldots,c_{k+j-1},c_{k+j}+1,c_{k+j+1},\right.$$
$$\left.\ldots,c_n;x_1,\ldots,x_n\right] ,$$

$j=1,\ldots,n-k$.

(5.4.10) $${}^{(k)}_{(2)}\phi^{(n)}_{AD}\left[a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;\ x_1,\ldots,x_n\right]$$

$$= {}^{(k)}_{(2)}\phi^{(n)}_{AD}\left[a,b_1,\ldots,b_k,b_{k+1},\ldots,b_{j+k-1},b_{j+k}-1,b_{j+k+1},\ldots,b_n;c_{k+1},\ldots,c_n;\right.$$
$$\left.x_1,\ldots,x_n\right]$$

$$+ \frac{x_{j+k}}{c_{j+k}}\ {}^{(k)}_{(2)}\phi^{(n)}_{AD}\left[a,b_1,\ldots,b_n;c_{k+1},\ldots,c_{k+j-1},c_{k+j}+1,c_{k+j+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

where $j = 1,\ldots,n-k$.

(5.4.11) $${}^{(k)}_{(2)}\phi^{(n)}_{AD}\left[a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a, b_1, \ldots, b_n; c_{k+1}, \ldots, c_{k+j-1}, c_{j+k}-1, c_{j+k+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$- \frac{x_{j+k}}{c_{j+k}(c_{j+k}-1)}\, {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a, b_1, \ldots, b_n; c_{k+1}, \ldots, c_{k+j-1}, c_{j+k}+1, c_{j+k+1}, \ldots, c_n; x_1, \ldots, x_n\right],$$

where $j = 1, \ldots, n-k$.

(5.4.12) $$(1+b_{j+k}-c_{j+k})\, {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a, b_1, \ldots, b_n; c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$= b_{j+k}\, {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a, b_1, \ldots, b_k, b_{k+1}, \ldots, b_{j+k-1}, b_{j+k}+1, b_{j+k+1}, \ldots, b_n; c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$+ (c_{j+k}-1)\, {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a, b_1, \ldots, b_n; c_{k+1}, \ldots, c_{j+k-1}, c_{j+k}+1, c_{j+k+1}, \ldots, c_n; x_1, \ldots, x_n\right],$$

where $j = 1, \ldots, n-k$.

(5.4.13) $${}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[a, b; c_1, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[a, b; c_1, \ldots, c_k, c_{k+1}, \ldots, c_{j+k-1}, c_{j+k}-1, c_{j+k+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$- \frac{x_{j+k}}{c_{j+k}(c_{j+k}-1)}\, {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[a, b; c_1, \ldots, c_k, c_{k+1}, \ldots, c_{j+k-1}, c_{j+k}+1, c_{j+k+1}, \ldots, c_n; x_1, \ldots, x_n\right],$$

where $j = 1, \ldots, n-k$.

$$(5.4.14)\quad {}^{(k)}_{(1)}\Phi^{(n)}_{D}[a,b_1,\ldots,b_n;c;\ x_1,\ldots,x_n]$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{D}[a,b_1,\ldots,b_n;c-1;x_1,\ldots,x_n] - \frac{a}{(c-1)}\Big\{x_1 b_1\, {}^{(k)}_{(1)}\Phi^{(n)}_{D}[a+1,b_1+1,\ldots,b_n;c+1;x_1,\ldots,x_n] + \ldots + x_k b_k\, {}^{(k)}_{(1)}\Phi^{(n)}_{D}[a+1,b_1,\ldots,b_k+1,b_{k+1},\ldots,b_n;c+1;x_1,\ldots,x_n]\Big\}$$

$$(5.4.15)\quad {}^{(k)}_{(3)}\Phi^{(n)}_{BD}[a,a_{k+1},\ldots,a_n,b_1,\ldots,b_k;c;x_1,\ldots,x_n]$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}[a,a_{k+1},\ldots,a_n,b_1,\ldots,b_k;c-1;x_1,\ldots,x_n] - \frac{a}{(c-1)}\Big\{x_1 b_1\, {}^{(k)}_{(3)}\Phi^{(n)}_{BD}[a+1,a_{k+1},\ldots,a_n,b_1+1,\ldots,b_k;c+1;x_1,\ldots,x_n] + \ldots + b_k x_k\, {}^{(k)}_{(3)}\Phi^{(n)}_{BD}[a+1,a_{k+1},\ldots,a_n,b_1,\ldots,b_k+1;c+1;x_1,\ldots,x_n]\Big\} - \frac{1}{(c-1)}\Big\{a_{k+1}x_{k+1}\, {}^{(k)}_{(3)}\Phi^{(n)}_{BD}[a,a_{k+1}+1,\ldots,a_n,b_1,\ldots,b_k;c+1;x_1,\ldots,x_n] + \ldots + a_n x_n\, {}^{(k)}_{(3)}\Phi^{(n)}_{BD}[a,a_{k+1},\ldots,a_n+1,b_1,\ldots,b_k,c+1;x_1,\ldots,x_n]\Big\},$$

5.5 GENERATING RELATIONS

Making an appeal to the definitions of the functions, we obtain the following generating relations :

$$(5.5.1)\quad (1-t)^{-a}\Big\{{}^{(k)}F^{(n)}_{CD}\Big[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_n}{1-t}\Big]\Big\}$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r\, t^r}{r!}\ {}^{(k)}F^{(n)}_{CD}[a+r,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n]$$

$$(5.5.2) \quad (1-t)^{-b}\left\{ {}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\frac{x_{k+1}}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r t^r}{r!} \; {}^{(k)}F^{(n)}_{CD}\left[a,b+r,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$(5.5.3) \quad (1-t)^{-b_i}\left\{ {}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,\frac{x_i}{1-t},\ldots,x_n\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r t^r}{r!} \; {}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_i+r,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right],$$

where $i = 1,\ldots,k$.

$$(5.5.4) \quad (1-t)^{-a}\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b;c,c_{k+1},\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r t^r}{r!} \; {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a+r,b;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$(5.5.5) \quad (1-t)^{-a}\left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r t^r}{r!} \; {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a+r,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$(5.5.6) \quad (1-t)^{-b}\left\{ {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\frac{x_{k+1}}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r t^r}{r!} \; {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b+r,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

$$(5.5.7)\qquad (1-t)^{-a}\left\{ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b,b_1,\dots,b_k;c;\frac{x_1}{1-t},\dots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r\, t^r}{r!}\ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a+r,b,b_1,\dots,b_k;c;x_1,\dots,x_n\right],$$

$$(5.5.8)\qquad (1-t)^{-b}\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b;c,c_{k+1},\dots,c_n;x_1,\dots,x_k,\frac{x_{k+1}}{1-t},\dots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty}\frac{(b)_r\, t^r}{r!}\ {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b+r;c,c_{k+1},\dots,c_n;x_1,\dots,x_n\right],$$

$$(5.5.9)\qquad (1-t)^{-b}\left\{ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b,b_1,\dots,b_k;c;x_1,\dots,x_k,\frac{x_{k+1}}{1-t},\dots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty}\frac{(b)_r\, t^r}{r!}\ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b+r,b_1,\dots,b_k;c;x_1,\dots,x_n\right],$$

$$(5.5.10)\qquad (1-t)^{-b_i}\left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,\frac{x_i}{1-t},\dots x_n\right]\right\}$$

$$= \sum_{r=0}^{\infty}\frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\dots,b_i+r,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n\right],$$

where $i = 1,\dots,k$.

$$(5.5.11)\qquad (1-t)^{-b_i}\left\{ {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,\frac{x_i}{1-t},\dots,x_n\right]\right\}$$

$$= \sum_{r=0}^{\infty}\frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[b,b_1,\dots,b_i+r,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n\right],$$

where $i = 1,\dots,k$.

(5.5.12) $$(1-t)^{-b_i}\left\{ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c;x_1,\ldots,\frac{x_i}{1-t},\ldots,x_k,\ldots,x_n\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_i+r,\ldots,b_k;c;x_1,\ldots,x_n\right],$$

where $i = 1, \ldots, k$.

(5.5.13) $$(1-t)^{-a}\left\{ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\, t^r}{r!}\ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a+r,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

(5.5.14) $$(1-t)^{-b}\left\{ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\frac{x_{k+1}}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\, t^r}{r!}\ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b+r,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

(5.5.15) $$(1-t)^{-b_i}\left\{ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,\frac{x_i}{1-t},\ldots,x_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_i+r,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right],$$

where $i = 1, \ldots, k$.

(5.5.16) $$(1-t)^{-a}\left\{ {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\, t^r}{r!}\ {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a+r,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

(5.5.17) $$(1-t)^{-b_i}\left\{{}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,\frac{x_i}{1-t},\ldots,x_k,\ldots,x_n\right]\right\}$$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r\,t^r}{r!}\;{}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_i+r,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right],$$

where $i=1,\ldots,k$.

(5.5.18) $$(1-t)^{-a}\left\{{}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r\,t^r}{r!}\;{}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a+r,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]$$

(5.5.19) $$(1-t)^{-b_i}\left\{{}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,\frac{x_i}{1-t},\ldots,x_n\right]\right\}$$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_i\,t^r}{r!}\;{}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a,b_1,\ldots,b_i+r,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right],$$

where $i=1,\ldots,k$.

(5.5.20) $$(1-t)^{-a}\left\{{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,a_{k+1},\ldots,a_n,b_1,\ldots,b_k;c;\frac{x_1}{1-t},\ldots,\frac{x_k}{1-t}\,x_{k+1},\ldots,x_n\right]\right\}$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r\,t^r}{r!}\;{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a+r,a_{k+1},\ldots,a_n,b_1,\ldots,b_k;c;x_1,\ldots,x_n\right]$$

(5.5.21) $$(1-t)^{-a_i}\left\{{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,a_{k+1},\ldots,a_n,b_1,\ldots,b_k;c;x_1,\ldots,x_k,x_{k+1},\ldots,\frac{x_i}{1-t},\ldots,x_n\right]\right\}$$

$$=\sum_{r=0}^{\infty}\frac{(a_i)_r\,t^r}{r!}\;{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,a_{k+1},\ldots,a_i+r,\ldots,a_n,b_1,\ldots,b_k;c;x_1,\ldots,x_n\right],$$

where $i=k+1,\ldots,n$.

$$(5.5.22) \qquad (1-t)^{-b_i} \left\{ {}^{(k)}_{(3)}\Phi^{(n)}_{BD} \left[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; c; x_1, \ldots, \frac{x_i}{1-t}, \ldots, x_k, x_{k+1}, \ldots, x_n \right] \right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r \, t^r}{r!} \; {}^{(k)}_{(3)}\Phi^{(n)}_{BD} \left[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_i+r, \ldots, b_k; c; x_1, \ldots, x_n \right],$$

where $i = 1, \ldots, k$.

$$(5.5.23) \qquad (1-t)^{-a} \left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{D} \left[a, b_1, \ldots, b_n; c; \frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t} \right] \right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r \, t^r}{r!} \; {}^{(k)}_{(1)}\Phi^{(n)}_{D} \left[a+r, b_1, \ldots, b_n; c; x_1, \ldots, x_n \right]$$

$$(5.5.24) \qquad (1-t)^{-b_i} \left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{D} \left[a, b_1, \ldots, b_n; c; x_1, \ldots, \frac{x_i}{1-t}, \ldots, x_n \right] \right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r \, t^r}{r!} \; {}^{(k)}_{(1)}\Phi^{(n)}_{D} \left[a, b_1, \ldots b_i+r, \ldots, b_n; c; x_1, \ldots, x_n \right],$$

where $i = 1, \ldots, n$.

$$(5.5.25) \qquad (1-t)^{-a} \left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{D} \left[a, b_1, \ldots, b_n; c; \frac{x_1}{1-t}, \ldots, \frac{x_k}{1-t}, x_{k+1}, \ldots, x_n \right] \right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r \, t^r}{r!} \; {}^{(k)}_{(2)}\Phi^{(n)}_{D} \left[a+r, b_1, \ldots, b_n; c; x_1, \ldots, x_n \right]$$

$$(5.5.26) \qquad (1-t)^{-b_i} \left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{D} \left[a, b_1, \ldots, b_n; c; x_1, \ldots, \frac{x_i}{1-t}, \ldots, x_n \right] \right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r \, t^r}{r!} \; {}^{(k)}_{(2)}\Phi^{(n)}_{D} \left[a, b_1, \ldots, b_i+r, \ldots, b_n; c; x_1, \ldots, x_n \right],$$

where $i = 1, \ldots, n$.

(5.5.27) $$(1-t)^{-a}\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[a,b,c_1,\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_k}{1-t},x_{k+1},\ldots,x_n\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\, t^r}{r!}\ {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[a+r,b,c_1,\ldots,c_n;x_1,\ldots,x_n\right],$$

(5.5.28) $$(1-t)^{-b}\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[a,b,c_1,\ldots,c_n;\frac{x_1}{1-t},\ldots,\frac{x_n}{1-t}\right]\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r\, t^r}{r!}\ {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[a,b+r;c_1,\ldots,c_n;x_1,\ldots,x_n\right].$$

In the next chapter **VI** , we shall make further study of these functions to derive their fractional integrations and integral representations .

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FRACTIONAL INTEGRATION AND INTEGRAL REPRESENTATIONS OF KARLSSON'S MULTIPLE HYPERGEOMETRIC FUNCTION & CONFLUENT FORMS OF CERTAIN GENERALIZED HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

CHAPTER VI

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FRACTIONAL INTEGRATION AND INTEGRAL REPRESENTATIONS OF KARLSSON'S MULTIPLE HYPERGEOMETRIC FUNCTION AND CONFLUENT FORMS OF CERTAIN GENERALIZED HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

6.1 **Introduction** Joshi [6] made use of the theory of fractional integral in order to deduce Eulerian Integral representations of hypergeometric functions of three variables, viz. H_A, H_B and H_C defined and studied by Srivastava (cf [10] to [14] see also [7] and [15]). In subsequent paper, Chandel [1] obtained similar integral representations for the multiple hypergeometric functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ of Lauricella [9] and also for their confluent forms $\Phi_2^{(n)}$ and $\Psi_2^{(n)}$. Further in another paper, Chandel [2] deduced similar integral representations for Exton's multiple hypergeometric functions ${}^{(k)}_{(1)}E_D^{(n)}$, and ${}^{(k)}_{(2)}E_D^{(n)}$ [5] related to Lauricella's $F_D^{(n)}$ and also for his own introduced multiple hypergeometric functions related to Lauricella $F_A^{(n)}$, $F_B^{(n)}$ and $F_C^{(n)}$ multiple hypergeometric functions

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From this chapter a paper entitled "Fractional integration and integral representations of Karlsson's multiple hypergeometric function and its Confluent forms" has been published in JÑĀNĀBHA, 20(1990), 101-110 .

Recently, Chandel and Gupta [3] have defined and studied multiple hypergeometric functions ${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$ and ${}^{(k)}F^{(n)}_{BD}$ related to Lauricella's functions [9]. Motivated by this paper Karlsson [8], also introduced fourth possible Lauricella function $F^{(n)}_{CD}$. Very recently Chandel and Vishwakarma [4] have studied confluent forms ${}^{(k)}_{(1)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(3)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(4)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(5)}\Phi^{(n)}_{CD}$ and ${}^{(k)}_{(6)}\Phi^{(n)}_{CD}$ of Karlsson's ${}^{(k)}F^{(n)}_{CD}$. Also in previous chapter 'V', we have introduced various confluent forms of certain generalized hypergeometric functions of several variables.

In the present chapter, we shall make use of the theory of fractional integration to derive Eulerian integral representations for multiple hypergeometric function ${}^{(k)}F^{(n)}_{CD}$ of Karlsson [8] and for various confluent forms of multiple hypergeometric functions of several variables introduced in chapter V.

We recall that the rule for fractional integration by parts is given by

$$(6.1.1) \qquad \int_a^b U \frac{\partial^\nu V}{\partial(b-x)^\nu}\, dx = \int_a^b V \frac{\partial^\nu U}{\partial(x-a)^\nu}\, dx .$$

If $Re(\nu) > 0$, the fractional derivatives occuring in (6.1.1) defined by the following integrals :

$$(6.1.2) \qquad \frac{\partial^{\nu} U}{\partial (x-a)^{\nu}} = \frac{1}{\Gamma(-\nu)} \int_a^x (x-y)^{-\nu-1}\, U(y)\, dy ,$$

$$(6.1.3) \qquad \frac{\partial^{\nu} V}{\partial (b-x)^{\nu}} = \frac{1}{\Gamma(-\nu)} \int_x^b (y-x)^{-\nu-1}\, V(y)\, dy .$$

If $Re(\nu) > 0$ and U,V are expressible in terms of the series

$$U = \sum A_r\, (x-a)^{\rho+r-1} \quad , \qquad V = \sum B_s\, (b-x)^{\rho+s-1} \quad ,$$

then the fractional derivatives are obtained by differentiating the above series term by term, with the help of the formula

$$(6.1.4) \qquad \frac{\partial^{\nu} W^{\mu-1}}{\partial W^{\nu}} = \frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}\, W^{\mu-\nu-1} ,$$

Provided $\mu \neq \nu$.

6.2 THE INTEGRAL REPRESENTATIONS

In this section, we derive the Eulerian integral representations for the functions ${}^{(k)}F^{(n)}_{CD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(3)}\Phi^{(n)}_{CD}$,

${}^{(k)}_{(4)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(5)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(6)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{AD}$, ${}^{(k)}_{(3)}\Phi^{(n)}_{BD}$, ${}^{(k)}_{(1)}\Phi^{(n)}_{D}$,

${}^{(k)}_{(2)}\Phi^{(n)}_{D}$ and ${}^{(k)}_{(1)}\Phi^{(n)}_{C}$.

Consider

$$\frac{\partial^{\beta_1-\nu_1+\cdots+\beta_k-\nu_k}}{\partial x_1^{\beta_1-\nu_1} \cdots \partial x_k^{\beta_k-\nu_k}} \Big\{ x_1^{\beta_1-1} \cdots x_k^{\beta_k-1}\, {}^{(k)}F^{(n)}_{CD} \left[\alpha, \nu, \nu_1, \ldots, \nu_k ; \gamma, \gamma_{k+1}, \ldots, \gamma_n ; x_1, \ldots, x_n \right] \Big\}$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_k}(\nu_1)_{m_1}\ldots(\nu_k)_{m_k}(\nu)_{m_{k+1}+\ldots+m_n}}{(\gamma)_{m_1+\ldots+m_k}(\gamma_{k+1})_{m_{k+1}}\ldots(\gamma_n)_{m_n}} \frac{1}{m_1!\ldots m_n!}$$

$$\frac{\partial^{\beta_1-\nu_1+\ldots+\beta_k-\nu_k}}{\partial x_1^{\beta_1-\nu_1}\ldots x_k^{\beta_k-\nu_k}} \left\{ x_1^{\beta_1+m_1-1}\ldots x_k^{\beta_k+m_k-1} . x_{k+1}^{m_{k+1}}\ldots x_n^{m_n} \right\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\beta_j)}{\Gamma(\nu_j)} (x_j)^{\nu_j-1} \, {}^{(k)}F^{(n)}_{CD}[\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n]$$

Hence by using the relation (6.1.2), we have

$${}^{(k)}F^{(n)}_{CD}[\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n]$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)} x_j^{1-\nu_j} \frac{\partial^{\beta_1-\nu_1+\ldots+\beta_k-\nu_k}}{\partial x_1^{\beta_1-\nu_1}\ldots x_k^{\beta_k-\nu_k}} \left\{ x_1^{\beta_1-1}\ldots x_k^{\beta_k-1} . \right.$$

$$\left. {}^{(k)}F^{(n)}_{CD}[\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots x_n] \right\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)(x_j)^{-\nu_j+1}}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)} \int_0^{x_1}\ldots\int_0^{x_k} (x_1-y_1)^{\nu_1-\beta_1-1} y_1^{\beta_1-1}\ldots(x_k-y_k)^{\nu_k-\beta_k-1}$$

$$y_k^{\beta_k-1} \, {}^{(k)}F^{(n)}_{CD}[\alpha, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; y_1, \ldots, y_k, x_{k+1}, \ldots, x_n] .$$

$$dy_1 \ldots dy_k .$$

Now putting every $y_j = x_j t_j$ where $j = 1, \ldots, k$.

we establish

$$(6.2.1)\quad {}^{(k)}F^{(n)}_{CD}\left[\alpha,\nu,\beta_1,\ldots,\beta_k;\gamma;\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n\right]$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\Gamma(\nu_j-\beta_j)} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} (1-t_i)^{\nu_i-\beta_i-1}\,(t_i)^{\beta_i-1}$$

$${}^{(k)}F^{(n)}_{CD}\left[\alpha,\nu,\nu_1,\ldots,\nu_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n\right] dt_1\ldots dt_k\,,$$

provided $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$, $i=1,\ldots,k$.

For brevity , we use the following operator Ω defined by Chandel [1,(2.2)]:

$$(6.2.2)\quad \Omega^{\beta_1,\ldots,\beta_k}_{\nu_1,\ldots,\nu_k}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} (t_i)^{\beta_i-1}\,(1-t_i)^{\nu_i-\beta_i-1} \left\{\ \right\}.$$

$$dt_1\ \ldots\ dt_k\ ,$$

where $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$, $i=1,\ldots,k$.

Hence the relation (6.2.1) can be written as

$$(6.2.3)\ \Omega\left\{{}^{(k)}F^{(n)}_{CD}\left[\alpha,\nu,\nu_1,\ldots,\nu_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}F^{(n)}_{CD}\left[\alpha,\nu,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n\right],$$

provided $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$, $i=1,\ldots,k$.

Applying the same techniques, we also obtain the following operational relationships :

$$(6.2.4)\ \Omega\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[\alpha,\nu_1,\ldots,\nu_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[\alpha,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n\right],$$

provided $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$, where $i=1,\ \ldots,\ k$.

$$(6.2.5)\ \Omega\left\{ {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[\nu,\nu_1,\ldots,\nu_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[\nu,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n\right],$$

provided $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$, where $i=1,\ \ldots,\ k$.

$$(6.2.6)\ \Omega\left\{ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[\alpha,\nu,\nu_1,\ldots,\nu_k;\gamma;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[\alpha,\nu,\beta_1,\ldots,\beta_k;\gamma;\ x_1,\ldots,x_n\right],$$

provided $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$, $i=1,\ \ldots\ k$.

$$(6.2.7)\ \Omega\left\{ {}^{(k)}F^{(n)}_{CD}\left[\alpha,\nu,\gamma_1,\nu_2,\ldots,\nu_k;\beta_1,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,x_1t_1x_2t_2,\ldots,\right.\right.$$
$$\left.\left. x_1t_1x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}F^{(n)}_{CD}\left[\alpha,\nu,\gamma_1,\beta_2,\ldots,\beta_k;\nu_1,\gamma_{k+1},\ldots,\gamma_n;x_1,x_1x_2,\ldots,x_1x_k,x_{k+1},\right.$$
$$\left.\ldots,x_n\right],$$

provided $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$, $i=1,\ \ldots,\ k$.

$$(6.2.8)\ \Omega\left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[\alpha,\gamma_1,\nu_2,\ldots,\nu_k;\beta_1,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,x_1t_1x_2t_2,\ldots,\right.\right.$$
$$\left.\left. x_1t_1x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[\alpha, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1 x_1, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n\right],$$

provided $0 < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i = 1, \ldots, k$.

$$(6.2.9)\quad \Omega\left\{ {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[\nu, \gamma_1, \nu_2, \ldots, \nu_k; \beta_1, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1, x_1 t_1 x_2 t_2, \ldots, x_1 t_1 x_k t_k, x_{k+1}, \ldots, x_n\right]\right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[\nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n\right],$$

provided $0 < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i = 1, \ldots, k$.

$$(6.2.10)\quad \Omega\left\{ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[\alpha, \nu, \gamma_1, \nu_2, \ldots, \nu_k; \beta_1; x_1 t_1, x_1 t_1 x_2 t_2, \ldots, x_1 t_1 x_k t_k, x_{k+1}, \ldots, x_n\right]\right\}$$

$$= {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[\alpha, \nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n\right],$$

provided $0 < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i = 1, \ldots, k$.

$$(6.2.11)\quad \Omega\left\{ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a, b, \nu_1, \ldots, \nu_k; \beta_{k+1}, \ldots, \beta_n; x_1 t_1, \ldots, x_n t_n\right]\right\}$$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a, b, \beta_1, \ldots, \beta_k; \nu_{k+1}, \ldots, \nu_n; x_1, \ldots, x_n\right],$$

provided $0 < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i = 1, \ldots, n$.

$$(6.2.12)\quad \Omega\left\{ {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a, \nu_1, \ldots, \nu_k; \beta_{k+1}, \ldots, \beta_n; x_1 t_1, \ldots, x_n t_n\right]\right\}$$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a, \beta_1, \ldots, \beta_k; \nu_{k+1}, \ldots, \nu_n; x_1, \ldots, x_n\right],$$

provided $0 < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i = 1, \ldots, n$.

$$(6.2.13)\ \Omega\left\{{}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,\nu_1,\ldots,\nu_k;c_{k+1},\ldots,c_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,\beta_1,\ldots,\beta_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right] ,$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,k$.

$$(6.2.14)\ \Omega\left\{{}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a,\nu_1,\ldots,\nu_k;c_{k+1},\ldots,c_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a,\beta_1,\ldots,\beta_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right] ,$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,k$.

$$(6.2.15)\ \Omega\left\{{}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a,\nu_1,\ldots,\nu_n;c_{k+1},\ldots,c_n;x_1t_1,\ldots,x_nt_n\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a,\beta_1,\ldots,\beta_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right] ,$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,n$.

$$(6.2.16)\ \Omega\left\{{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,a_{k+1},\ldots,a_n,\nu_1,\ldots,\nu_k;c;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n\right]\right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,a_{k+1},\ldots,a_n,\beta_1,\ldots,\beta_k;c;\ x_1,\ldots,x_n\right] ,$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\beta_j)$, $j=1,\ldots,k$.

$$(6.2.17)\ \Omega\left\{{}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a,\nu_1,\ldots,\nu_n;c;\ x_1t_1,\ldots,x_nt_n\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a,\beta_1,\ldots,\beta_n;c;\ x_1,\ldots,x_n\right] ,$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j = 1, \ldots, n$.

(6.2.18) $\Omega \left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{D} [a, \nu_1, \ldots, \nu_n; c; x_1 t_1, \ldots, x_n t_n] \right\}$

$= {}^{(k)}_{(2)}\Phi^{(n)}_{D} [a, \beta_1, \ldots, \beta_n; c; x_1, \ldots, x_n]$,

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j = 1, \ldots, n$.

(6.2.19) $\Omega \left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{C} [a, b; \beta_1, \ldots, \beta_n; x_1 t_1, \ldots, x_n t_n] \right\}$

$= {}^{(k)}_{(1)}\Phi^{(n)}_{C} [a, b; \nu_1, \ldots, \nu_n; x_1, \ldots, x_n]$,

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j = 1, \ldots, n$.

(6.2.20) $\Omega \left\{ {}^{(k)}_{(5)}\Phi^{(n)}_{CD} [a, b, b_1, \ldots, b_k; \beta_{k+1}, \ldots, \beta_n; x_1, \ldots, x_k, x_{k+1} t_{k+1}, \ldots, x_n t_n] \right\}$

$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD} [a, b, b_1, \ldots, b_k; \nu_{k+1}, \ldots, \nu_n; x_1, \ldots, x_n]$,

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j = k+1, \ldots, n$.

(6.2.21) $\Omega \left\{ {}^{(k)}_{(6)}\Phi^{(n)}_{CD} [a, b_1, \ldots, b_k; \beta_{k+1}, \ldots, \beta_n; x_1, \ldots, x_k, x_{k+1} t_{k+1}, \ldots, x_n t_n] \right\}$

$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD} [a, b_1, \ldots, b_k; \nu_{k+1}, \ldots, \nu_n; x_1, \ldots, x_n]$,

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j = k+1, \ldots, n$.

(6.2.22) $\Omega \left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{AD} [a, b_1, \ldots, b_n; \beta_{k+1}, \ldots, \beta_n; x_1, \ldots, x_k, x_{k+1} t_{k+1}, \ldots, x_n t_n] \right\}$

$= {}^{(k)}_{(2)}\Phi^{(n)}_{AD} [a, b_1, \ldots, b_n; \nu_{k+1}, \ldots, \nu_n; x_1, \ldots, x_n]$,

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j = k+1, \ldots, n$.

(6.2.23) $\Omega\left\{{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,\nu_{k+1},\ldots,\nu_n,b_1,\ldots,b_k;c;x_1,\ldots,x_k,x_{k+1}t_{k+1},\ldots,x_nt_n\right]\right\}$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,\beta_{k+1},\ldots,\beta_n,b_1,\ldots,b_k;c;x_1,\ldots,x_n\right],$$

provided $0<Re(\beta_j)<Re(\nu_j)$, $j = k+1, \ldots, n$.

(6.2.24) $\Omega\left\{{}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[\nu_1,b,b_1,\nu_2,\ldots,\nu_k;\beta_{k+1},\ldots,\beta_n;x_1t_1,x_1t_1x_2t_2,\ldots,x_1t_1x_nt_n\right]\right\}$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[\beta_1,b,b_1,\beta_2,\ldots,\beta_k;\nu_{k+1},\ldots,\nu_n;\ x_1,x_1x_2,\ldots,x_1x_n\right],$$

provided $0<Re(\beta_j)<Re(\nu_j)$, $j = 1, \ldots, n$.

which suggests k results in the following unified form :

(6.2.25) $\Omega\left\{{}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[\nu_i,b,\nu_1,\ldots,\nu_{i-1},b_i,\nu_{i+1},\ldots,\nu_k;\beta_{k+1},\ldots,\beta_n;x_1t_1x_it_i,\ldots,x_{i-1}t_{i-1}x_it_i,x_it_i,x_{i+1}t_{i+1}x_it_i,\ldots,x_nt_nx_it_i\right]\right\}$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[\beta_i,b,\beta_1,\ldots,\beta_{i-1},b_i,\beta_{i+1},\ldots,\beta_k;\nu_{k+1},\ldots,\nu_n;x_1x_i,\ldots,x_{i-1}x_i,x_i,x_ix_{i+1},\ldots,x_nx_i\right],$$

provided $0<Re(\beta_j)<Re(\nu_j)$, $j=1, \ldots, n$ and $i = 1, \ldots, k$.

(6.2.26) $\Omega\left\{{}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[\nu_1,b_1,\nu_2,\ldots,\nu_k;\beta_{k+1},\ldots,\beta_n;x_1t_1,x_1t_1x_2t_2,\ldots,x_1t_1x_nt_n\right]\right\}$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[\beta_1,b_1,\beta_2,\ldots,\beta_k;\nu_{k+1},\ldots,\nu_n;x_1,x_1x_2,\ldots,x_1x_n\right],$$

provided $0<Re(\beta_j)<Re(\nu_j)$, $j = 1, \ldots, n$.

which suggests k results in the following unified form :

$$(6.2.27)\quad \Omega\left\{{}^{(k)}_{(6)}\phi^{(n)}_{CD}\left[\nu_i,\nu_1,\ldots,\nu_{i-1},b_i,\nu_{i+1},\ldots,\nu_k;\beta_{k+1},\ldots,\beta_n;x_1t_1x_it_i,\ldots,x_{i-1}t_{i-1}x_it_i,x_it_i,x_{i+1}t_{i+1}x_it_i,\ldots,x_nt_nx_it_i\right]\right\}$$

$$= {}^{(k)}_{(6)}\phi^{(n)}_{CD}\left[\beta_i,\beta_1,\ldots,\beta_{i-1},b_i,\beta_{i+1},\ldots,\beta_k;\nu_{k+1},\ldots,\nu_n;x_1x_i,\ldots,x_{i-1}x_i,x_i,x_{i+1}x_i,\ldots,x_nx_i\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,n$ and $i=1,\ldots,k$.

$$(6.2.28)\quad \Omega\left\{{}^{(k)}_{(2)}\phi^{(n)}_{AD}\left[\nu_1,b_1,\nu_2,\ldots,\nu_n;c_{k+1},\ldots,c_n;x_1t_1,x_1t_1x_2t_2,\ldots,x_1t_1x_nt_n\right]\right\}$$

$$= {}^{(k)}_{(2)}\phi^{(n)}_{AD}\left[\beta_1,b_1,\beta_2,\ldots,\beta_n;c_{k+1},\ldots,c_n;x_1,x_1x_2,\ldots,x_1x_n\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,n$.

which suggests n results in the following unfied form :

$$(6.2.29)\quad \Omega\left\{{}^{(k)}_{(2)}\phi^{(n)}_{AD}\left[\nu_i,\nu_1,\ldots,\nu_{i-1},b_i,\nu_{i+1},\ldots,\nu_n;c_{k+1},\ldots,c_n;x_1x_it_1t_i,\ldots,x_{i-1}t_{i-1}x_it_i,x_it_i,x_{i+1}t_{i+1}x_it_i,\ldots,x_nt_nx_it_i\right]\right\}$$

$$= {}^{(k)}_{(2)}\phi^{(n)}_{AD}\left[\beta_i,\beta_1,\ldots,\beta_{i-1},b_i,\beta_{i+1},\ldots,\beta_n;c_{k+1},\ldots,c_n;x_1x_i,\ldots,x_{i-1}x_i,x_i,x_{i+1}x_i,\ldots,x_nx_i\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,n$ and $i=1,\ldots,n$.

$$(6.2.30)\quad \Omega\left\{{}^{(k)}_{(1)}\phi^{(n)}_{D}\left[\nu_1,b_1,\nu_2,\ldots,\nu_n;c;x_1t_1,x_1t_1x_2t_2,\ldots,x_1t_1x_nt_n\right]\right\}$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{D}\left[\beta_1,b_1,\beta_2,\ldots,\beta_n;c;\ x_1,x_1x_2,\ldots,x_1x_n\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,n$.

which suggests **n** results in the following unified form :

$$(6.2.31)\quad \Omega\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[\nu_i, \nu_1, \ldots, \nu_{i-1}, b_i, \nu_{i+1}, \ldots, \nu_n; c;\ x_1 t_1 x_i t_i, \ldots, x_{i-1} t_{i-1} x_i t_i, x_i t_i, x_{i+1} t_{i+1} x_i t_i, \ldots, x_n t_n x_i t_i\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[\beta_i, \beta_1, \ldots, \beta_{i-1}, b_i, \beta_{i+1}, \ldots, \beta_n; c; x_1 x_i, \ldots, x_{i-1} x_i, x_i, x_{i+1} x_i, \ldots, x_n x_i\right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j=1, \ldots, n$ and $i=1, \ldots, n$.

$$(6.2.32)\quad \Omega\left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[a, b_1, \nu_2, \ldots, \nu_n; \beta_1; x_1 t_1, x_1 t_1 x_2 t_2, \ldots, x_1 t_1 x_n t_n\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[a, b_1, \beta_2, \ldots, \beta_n; \nu_1; x_1, x_1 x_2, \ldots, x_1 x_n\right],$$

$0 < Re(\beta_j) < Re(\nu_j)$, $j = 1, \ldots, n$.

which suggests **n** results in the following unified form :

$$(6.2.33)\quad \Omega\left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[a, \nu_1, \ldots, \nu_{i-1}, b_i, \nu_{i+1}, \ldots, \nu_n; \beta_i;\ x_1 t_1 x_i t_i, \ldots, x_{i-1} t_{i-1} x_i t_i, x_i t_i, x_{i+1} t_{i+1} x_i t_i, \ldots, x_n t_n x_i t_i\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[a, \beta_1, \ldots, \beta_{i-1}, b_i, \beta_{i+1}, \ldots, \beta_n; \nu_i; x_1 x_i, \ldots, x_{i-1} x_i, x_i, x_{i+1} x_i, \ldots, x_n x_i\right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j=1, \ldots, n$ and $i=1, \ldots, n$.

$$(6.2.34)\quad \Omega\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[\nu_1, a, c_1; \beta_2, \ldots, \beta_n; x_1 t_1, x_1 t_1 x_2 t_2, \ldots, x_1 t_1 x_n t_n\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[\beta_1 a; c_1; \nu_2, \ldots, \nu_n; x_1, x_1 x_2, \ldots, x_1 x_n\right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j=1, \ldots, n$.

which suggests n results in the following unified form :

$$(6.2.35)\quad \Omega\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[\nu_i, a; \beta_1, \ldots, \beta_{i-1}, c_i, \beta_{i+1}, \ldots, \beta_n; x_1x_it_1t_i, \ldots, x_{i-1}t_{i-1}x_it_i, x_it_i, x_{i+1}t_{i+1}x_it_i, \ldots, x_nt_nx_it_i\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[\beta_i, a; \nu_1, \ldots, \nu_{i-1}, c_i, \nu_{i+1}, \ldots, \nu_n; x_1x_i, \ldots, x_{i-1}x_i, x_i, x_{i+1}x_i, \ldots, x_nx_i\right],$$

$0 < Re(\beta_j) < Re(\nu_j)$, $j=1, \ldots, n$ and $i=1, \ldots, n$.

$$(6.2.36)\quad \Omega\left\{ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a, \nu_{k+1}, b_1, \ldots, b_k; c_{k+1}, \beta_{k+2}, \ldots, \beta_n; x_1t_1, \ldots, x_kt_k, x_{k+1}t_{k+1}, x_{k+2}t_{k+2}x_{k+1}t_{k+1}, \ldots, x_nt_nx_{k+1}t_{k+1}\right]\right\}$$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a, \beta_{k+1}, b_1, \ldots, b_k; c_{k+1}, \nu_{k+2}, \ldots, \nu_n; x_1, \ldots, x_k, x_{k+1}, x_{k+1}x_{k+2}, \ldots, x_{k+1}x_n\right],$$

$0 < Re(\beta_j) < Re(\nu_j)$, $j = k+1, \ldots, n$.

which suggests n-k results in the following unified form :

$$(6.2.37)\quad \Omega\left\{ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a, \nu_i, b_1, \ldots, b_k; \beta_{k+1}, \ldots, \beta_{i-1}, c_i, \beta_{i+1}, \ldots, \beta_n; x_1t_1, \ldots, x_kt_k, x_{k+1}t_{k+1}x_it_i, \ldots, x_{i-1}t_{i-1}x_it_i, x_it_i, x_{i+1}t_{i+1}x_it_i, \ldots, x_nt_nx_it_i\right]\right\}$$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a, \beta_i, b_1, \ldots, b_k; \nu_{k+1}, \ldots, \nu_{i-1}, c_i, \nu_{i+1}, \ldots, \nu_n; x_1, \ldots, x_k, x_ix_{k+1}, \ldots, x_{i-1}x_i, x_i, x_{i+1}x_i, \ldots, x_nx_i\right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j=k+1, \ldots, n$ and $i=k+1, \ldots, n$.

(6.2.38) $\Omega\left\{ {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[\nu_1, a_{k+1}, \ldots, a_n, b_1, \nu_2, \ldots, \nu_k; c;\ x_1 t_1,\ x_1 t_1 x_2 t_2,\right.\right.$

$\left.\left.\ldots, x_1 t_1 x_k t_k, x_{k+1} t_{k+1}, \ldots, x_n t_n\right]\right\}$

$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[\beta_1, a_{k+1}, \ldots, a_n, b_1, \beta_2, \ldots, \beta_k; c;\ x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n\right]$,

$0 < Re(\beta_j) < Re(\nu_j)$, $j = 1, \ldots, k$.

which suggests **k** results in the following unified form :

(6.2.39) $\Omega\left\{ {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[\nu_i, a_{k+1}, \ldots, a_n, \nu_1, \ldots, \nu_{i-1}, b_i, \nu_{i+1}, \ldots, \nu_k; c; x_1 t_1 x_i t_i,\right.\right.$

$\left.\left.\ldots, x_{i-1} t_{i-1} x_i t_i,\ x_i t_i,\ x_{i+1} t_{i+1} x_i t_i, \ldots, x_k t_k x_i t_i, x_{k+1} t_{k+1}, \ldots, x_n t_n\right]\right\}$

$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[\beta_i, a_{k+1}, \ldots, a_n, \beta_1, \ldots, \beta_{i-1}, b_i, \beta_{i+1}, \ldots, \beta_k; c;\ x_1 x_i, \ldots, x_{i-1} x_i, x_i,\right.$

$\left. x_{i+1} x_i, \ldots, x_k x_i, x_{k+1}, \ldots, x_n\right]$,

$0 < Re(\beta_j) < Re(\nu_j)$, $j=1, \ldots, k$ and $i = 1, \ldots, k$.

(6.2.40) $\Omega\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a, b_1, \nu_2, \ldots, \nu_k, b_{k+1}, \ldots, b_n; \beta_1; x_1 t_1, x_1 t_1 x_2 t_2, \ldots,\right.\right.$

$\left.\left. x_1 t_1 x_k t_k, x_{k+1} t_{k+1}, \ldots, x_n t_n\right]\right\}$

$= {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a, b_1, \beta_2, \ldots, \beta_k, b_{k+1}, \ldots, b_n; \nu_1; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n\right]$,

provided $0 < Re(\beta_j) < Re(\nu_j)$, $j=1, \ldots, k$.

which suggests k results in the following unified form :

(6.2.41) $\Omega\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a, \nu_1, \ldots, \nu_{i-1}, b_i, \nu_{i+1}, \ldots, \nu_k, b_{k+1}, \ldots, b_n; \beta_i; x_1 t_1 x_i t_i,\right.\right.$

$\left.\left.\ldots, x_{i-1} t_{i-1} x_i t_i,\ x_i t_i, x_{i+1} t_{i+1} x_i t_i, \ldots, x_k t_k x_i t_i, x_{k+1} t_{k+1}, \ldots, x_n t_n\right]\right\}$

$= {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a, \beta_1, \ldots, \beta_{i-1}, b_i, \beta_{i+1}, \ldots, \beta_k, b_{k+1}, \ldots, b_n; \nu_i;\ x_1 x_i, \ldots, x_{i-1} x_i,\ x_i,\right.$

$\left. x_{i+1} x_i, \ldots,\ x_k x_i, x_{k+1}, \ldots, x_n\right]$,

provided $0 < \mathrm{Re}(\beta_j) < \mathrm{Re}(\nu_j)$, $j=1,\ldots,k$ and $i=1,\ldots,k$.

(6.2.42) $\Omega\left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[\nu_1, b_1, \nu_2, \ldots, \nu_k, b_{k+1}, \ldots, b_n; c; x_1 t_1, x_1 t_1 x_2 t_2, \ldots, x_1 t_1 x_k t_k, x_{k+1} t_{k+1}, \ldots, x_n t_n\right]\right\}$

$= {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[\beta_1, b_1, \beta_2, \ldots, \beta_k, b_{k+1}, \ldots, b_n; c; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n\right]$,

$0 < \mathrm{Re}(\beta_j) < \mathrm{Re}(\nu_j)$, $j=1,\ldots,k$.

which suggests k results in the following unfied form :

(6.2.43) $\Omega\left\{ {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[\nu_i, \nu_1, \ldots, \nu_{i-1}, b_i, \nu_{i+1}, \ldots, \nu_k, b_{k+1}, \ldots, b_n; c; x_1 t_1 x_i t_i, \ldots, x_{i-1} t_{i-1} x_i t_i, x_i t_i, x_{i+1} t_{i+1} x_i t_i, \ldots, x_k t_k x_i t_i, x_{k+1} t_{k+1}, \ldots, x_n t_n\right]\right\}$

$= {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[\beta_i, \beta_1, \ldots, \beta_{i-1}, b_i, \beta_{i+1}, \ldots, \beta_k, b_{k+1}, \ldots, b_n; c; x_1 x_i, \ldots, x_{i-1} x_i, x_i, x_{i+1} x_i, \ldots, x_k x_i, x_{k+1}, \ldots, x_n\right]$,

provided $0 < \mathrm{Re}(\beta_j) < \mathrm{Re}(\nu_j)$, $j=1, \ldots, k$ and $i=1, \ldots, k$.

(6.2.44) $\Omega\left\{ {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[b, \nu_1; c_1, \beta_2, \ldots, \beta_k, c_{k+1}, \ldots, c_n; x_1 t_1, x_1 t_1 x_2 t_2, \ldots, x_1 x_k t_1 t_k, x_{k+1} t_{k+1}, \ldots, x_n t_n\right]\right\}$

$= {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[b, \beta_1; c_1, \nu_2, \ldots, \nu_k, c_{k+1}, \ldots, c_n; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n\right]$,

provided $0 < \mathrm{Re}(\beta_j) < \mathrm{Re}(\nu_j)$, $j = 1, \ldots, k$,

which suggests k results in the following unified form :

(6.2.15) $\Omega\left\{{}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[b,\nu_i;\beta_1,\ldots,\beta_{i-1};c_i,\beta_{i+1},\ldots,\beta_k,c_{k+1},\ldots,c_n;x_1t_1x_it_i,\right.\right.$

$\left.\ldots,x_{i-1}t_{i-1}x_it_i,x_it_i,x_{i+1}t_{i+1}x_it_i,\ldots,x_kt_kx_it_i,x_{k+1}t_{k+1},\ldots,x_nt_n\right]$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[b,\beta_i;\ \nu_1,\ldots,\nu_{i-1};c_i,\nu_{i+1},\ldots,\nu_k,c_{k+1},\ldots,c_n;\ x_1x_i,\ldots,x_{i-1}x_i\ ,\ x_i,\right.$$
$$\left.x_{i+1}x_i,\ldots,x_kx_i,x_{k+1},\ldots,x_n\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,k$ and $i=1,\ldots,k$.

6.3 FURTHER EXTENSIONS

In this present section, we derive extensions of the results obtained in the previous section 6.2 to hold for multiple hypergeometric function defined by

$$\tilde{F}(x_1,\ldots,x_n) = F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 1 & (-;\nu) & \\ k & \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k & x_1,\ldots,x_n \\ 0 & \ldots\ \ldots\ \ldots & \\ 1 & (\gamma;-) & \\ n & \mu_1;\ldots;\mu_k;\gamma_{k+1},\ldots,\gamma_n & \end{array}\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n}(\nu)_{m_{k+1}+\ldots+m_n}(\nu_1)_{m_1}\ldots(\nu_k)_{m_k}(\lambda_1)_{m_1}\ldots(\lambda_k)_{m_k}}{(\gamma)_{m_1+\ldots+m_k}(\mu_1)_{m_1}\ldots(\mu_k)_{m_k}(\gamma_{k+1})_{m_{k+1}}\ldots(\gamma_n)_{m_n}} \cdot \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

From the above definition it follows that

$$\frac{\partial^{\mu_1-\lambda_1+\ldots+\mu_k-\lambda_k}}{\partial t_1^{\mu_1-\lambda_1}\ldots\partial t_k^{\mu_k-\lambda_k}}\left\{t_1^{\mu_1-1}\ldots t_k^{\mu_k-1}\tilde{F}(x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n)\right\}$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\mu_j)}{\Gamma(\mu_j)}\, t_j^{\lambda_j-1}\,{}^{(k)}F_{CD}^{(n)}[\alpha,\nu,\nu_1,\ldots,\nu_k:\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,\ldots,x_kt_k,$$
$$x_{k+1},\ldots,x_n].$$

Using the relation (6.2.3) we have

$${}^{(k)}F_{CD}^{(n)}[\alpha,\nu,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;\ x_1,\ldots,x_n]$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)\,\Gamma(\mu_j)}\int_0^1\ldots\int_0^1\prod_{i=1}^{k}(t_i)^{\beta_i-\lambda_i}\,(1-t_i)^{\nu_i-\beta_i-1}\,.$$

$$\frac{\partial^{\mu_1-\lambda_1+\ldots+\mu_k-\lambda_k}}{\partial t_1^{\mu_1-\lambda_1}\ldots t_k^{\mu_k-\lambda_k}}\left\{t_1^{\mu_1-1}\ldots t_k^{\mu_k-1}\tilde{F}(x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n)\right\}dt_1\ldots dt_k\,.$$

Now making an appeal to the result (6.1.1), we derive

$$(6.3.1)\quad {}^{(k)}F_{CD}^{(n)}[\alpha,\nu,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n]$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\Gamma(\mu_j)\Gamma(\nu_j-\beta_j)}\int_0^1\ldots\int_0^1 t_1^{\mu_1-1}\ldots t_k^{\mu_k-1}\,{}^{(k)}F_{CD}^{(n)}[x_1t_1,\ldots,x_kt_k,x_{k+1},$$
$$\ldots,x_n]\,.\,\frac{\partial^{\mu_1-\lambda_1+\ldots+\mu_k-\lambda_k}}{\partial(1-t_1)^{\mu_1-\lambda_1}\ldots\partial(1-t_k)}\left\{\prod_{i=1}^{k}(t_i)^{\beta_i-\lambda_i}\,(1-t_i)^{\nu_i-\beta_i-1}\right\}dt_1\ldots dt_k$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\Gamma(\mu_j)\Gamma(\nu_j+\lambda_j-\mu_j-\beta_j)} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} (t_i)^{\mu_i-1} (1-t_i)^{\nu_i+\lambda_i-\mu_i-\beta_i-1} .$$

$${}_2F_1 [\nu_i-\beta_i, \lambda_i-\beta_i; \nu_i+\lambda_i-\mu_i-\beta_i; 1-t_i]\, \tilde{F}(x_1t_1, \ldots, x_kt_k, x_{k+1}, \ldots, x_n)\,dt_1 \ldots dt_k .$$

Here for brevity, we use the following operator R introduced by Chandel [1,(3.1)] :

$$(6.3.2)\quad R\{\ \} = \prod_{j=1}^{k} \frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\Gamma(\mu_j)\Gamma(\nu_j+\lambda_j-\beta_j-\mu_j)} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} (t_i)^{\mu_i-1} .$$

$$(1-t_i)^{\nu_i+\lambda_i-\mu_i-\beta_i-1}\ {}_2F_1[\nu_i-\beta_i, \lambda_i-\beta_i; \nu_i+\lambda_i-\mu_i-\beta_i; 1-t_i]\{\ \}dt_1\cdots dt_k .$$

and establish the result

$$(6.3.3)\quad R\left\{ F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 1 & (-;\nu) & \\ k & \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k & \\ 0 & \ldots\quad\ldots\quad\ldots & x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n \\ 1 & (\gamma;-) & \\ n & \mu_1;\ldots;\mu_k;\gamma_{k+1};\ldots;\gamma_n & \end{array}\right]\right\}$$

$$= {}^{(k)}F^{(n)}_{CD} [\alpha, \nu, \beta_1, \beta_2, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n] ,$$

provided $0 < Re(\mu_j) < Re(\nu_j+\lambda_j-\beta_j)$; $0 < Re(\beta_j) < Re(\nu_j)$, $j = 1,\ldots,k$.

[Extension of (6.2.3)]

Applying the same techniques, we obtain the following results :

$$(6.3.4)\quad R\left\{ F \begin{array}{c} 1 \\ 0 \\ k \\ 0 \\ 1 \\ n \end{array} \left[\begin{array}{c|c} \alpha & \\ \cdots \quad \cdots \quad \cdots & \\ \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k & x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n \\ \cdots \quad \cdots \quad \cdots & \\ (\gamma;-) & \\ \mu_1;\ldots;\mu_k;\gamma_{k+1};\ldots;\gamma_n & \end{array} \right] \right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[\alpha,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n\right],$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$; $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=1,\ldots,k$.

[Extension of (6.2.4)]

$$(6.3.5)\quad R\left\{ F \begin{array}{c} 0 \\ 1 \\ k \\ 0 \\ 1 \\ n \end{array} \left[\begin{array}{c|c} \cdots \quad \cdots \quad \cdots & \\ (-;\nu) & \\ \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k & x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n \\ \cdots \quad \cdots \quad \cdots & \\ (\gamma;-) & \\ \mu_1;\ldots;\mu_k;\gamma_{k+1};\ldots;\gamma_n & \end{array} \right] \right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[\nu,\beta_1,\ldots,\beta_k;\gamma;\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n\right],$$

valid if $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$; $0<\mathrm{Re}(\mu_i)<\mathrm{Re}(\nu_i+\lambda_i-\beta_i)$, $i=1,\ldots,k$.

[Extension Of (6.2.5)]

$$(6.3.6)\quad R\left\{ F \begin{array}{c} 1 \\ 1 \\ k \\ 0 \\ 1 \\ k \end{array} \left[\begin{array}{c|c} \alpha & \\ (-;\nu) & \\ \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k & x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n \\ \cdots \quad \cdots \quad \cdots & \\ (\gamma;-) & \\ \mu_1;\ldots;\mu_k & \end{array} \right] \right\}$$

$$= {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma; x_1, \ldots, x_n\right],$$

valid if $0 < \mathrm{Re}(\mu_i) < \mathrm{Re}(\nu_i + \lambda_i - \beta_i)$; $0 < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i=1, \ldots, k$.

[Extension of (6.2.6)]

$$(6.3.7)\ R\left\{ F\left[\begin{array}{l|c|l} 1 & \alpha & \\ 1 & (\lambda_1;\nu) & \\ k & \gamma_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k & x_1t_1, x_1t_1x_2t_2, \ldots, x_1t_1x_kt_k, \\ 0 & \ldots\quad\ldots\quad\ldots & \\ 1 & (\nu_1,\mu_1;-) & x_{k+1}, \ldots, x_n \\ n-1 & \mu_2;\ldots;\mu_k;\gamma_{k+1};\ldots;\gamma_n & \end{array}\right]\right\}$$

$$= {}^{(k)}F^{(n)}_{CD}\left[\alpha, \nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots, x_n\right],$$

provided $0 < \mathrm{Re}(\mu_j) < \mathrm{Re}(\nu_j + \lambda_j - \beta_j)$; $0 < \mathrm{Re}(\beta_j) < \mathrm{Re}(\nu_j)$, $j=1, \ldots, k$.

[Extension of (6.2.7)]

$$(6.3.8)\ R\left\{ F\left[\begin{array}{l|c|l} 1 & \alpha & \\ 1 & (\lambda_1;-) & \\ k & \gamma_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k & x_1t_1, x_1t_1x_2t_2, \ldots, x_1t_1x_kt_k, \\ 0 & \ldots\quad\ldots\quad\ldots & \\ 1 & (\beta_1,\mu_1;-) & x_{k+1}, \ldots, x_n \\ n-1 & \mu_2;\ldots;\mu_k;\gamma_{k+1};\ldots;\gamma_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[\alpha, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots, x_n\right],$$

provided $0 < \mathrm{Re}(\mu_j) < \mathrm{Re}(\nu_j + \lambda_j - \beta_j)$; $0 < \mathrm{Re}(\beta_j) < \mathrm{Re}(\nu_j)$, $j=1, \ldots, k$.

[Extension of (6.2.8)]

$$(6.3.9)\quad R\left\{F\left[\begin{array}{c|c|l} 0 & \cdots\quad\cdots\quad\cdots & \\ 1 & (\lambda_1,\nu) & \\ k & \gamma_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k & x_1t_1,\ x_1t_1x_2t_2,\ldots,x_1t_1x_kt_k, \\ 0 & \cdots\quad\cdots\quad\cdots & x_{k+1},\ldots,x_n \\ 1 & (\beta_1,\mu_1;-) & \\ n-1 & \mu_2;\ldots;\mu_k;\gamma_{k+1};\ldots;\gamma_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[\nu,\gamma_1,\beta_2,\ldots,\beta_k;\nu_1,\gamma_{k+1},\ldots,\gamma_n;x_1,x_1x_2,\ldots,x_1x_k,x_{k+1},\ldots,x_n\right]$$

provided $0<\mathrm{Re}(\beta_i)<\mathrm{Re}(\nu_i)$; $0<\mathrm{Re}(\mu_i)<\mathrm{Re}(\nu_i+\lambda_i-\beta_i)$ $i=1,\ldots,k$.

[Extension of (6.2.9)]

$$(6.3.10)\quad R\left\{F\left[\begin{array}{c|c|l} 1 & \alpha & \\ 1 & (\lambda_1;\nu) & \\ k & \gamma_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k & x_1t_1,x_1t_1x_2t_2,\ldots,x_1t_1x_kt_k, \\ 0 & \cdots\quad\cdots\quad\cdots & x_{k+1},\ldots,x_n \\ 1 & (\beta_1,\mu_1;-) & \\ k-1 & \mu_2;\ldots;\mu_k & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[\alpha,\nu,\gamma_1,\beta_2,\ldots,\beta_k;\nu_1;x_1,x_1x_2,\ldots,x_1x_k,x_{k+1},\ldots,x_n\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$; $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=1,\ldots,k$.

[Extension of (6.2.10)]

$$(6.3.11)\quad R\left\{F\left[\begin{array}{c|c|l} 1 & a & \\ 1 & (-;b) & \\ n & \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k;\lambda_{k+1};\ldots;\lambda_n & x_1t_1,\ldots,x_nt_n \\ 0 & \cdots\quad\cdots\quad\cdots & \\ 0 & \cdots\quad\cdots\quad\cdots & \\ n & \mu_1;\ldots;\mu_k;\mu_{k+1},\beta_{k+1};\ldots;\mu_n,\beta_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,\beta_1,\ldots,\beta_k;\nu_{k+1},\ldots,\nu_n;\ x_1,\ldots,x_n\right],$$

$$0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)\ ,\quad 0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)\ ,\quad j=1,\ldots,n\ .$$

[Extension of (6.2.11)]

$$(6.3.12)\qquad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 0 & \ldots\quad\ldots\quad\ldots & \\ n & \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k;\lambda_{k+1};\ldots;\lambda_n & x_1t_1,\ldots,x_nt_n \\ 0 & \ldots\quad\ldots\quad\ldots & \\ 0 & \ldots\quad\ldots\quad\ldots & \\ n & \mu_1;\ldots;\mu_k;\mu_{k+1},\beta_{k+1};\ldots;\mu_n,\beta_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a,\beta_1,\ldots,\beta_k;\nu_{k+1},\ldots,\nu_n;\ x_1,\ldots,x_n\right]\ ,$$

provided $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,n$.

[Extension of (6.2.12)]

$$(6.3.13)\qquad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 1 & (-;b) & \\ k & \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k & x_1t_1,\ldots,x_kt_k,x_{k+1}, \\ 0 & \ldots\quad\ldots\quad\ldots & \ldots,x_n \\ 0 & \ldots\quad\ldots\quad\ldots & \\ n & \mu_1;\ldots;\mu_k;c_{k+1};\ldots;c_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[a,b,\beta_1,\ldots,\beta_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right]\ ,$$

provided $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $j=1,\ldots,k$.

[Extension of (6.2.13)]

$$(6.3.14)\qquad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 0 & \ldots\quad\ldots\quad\ldots & \\ k & \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k & x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n \\ 0 & \ldots\quad\ldots\quad\ldots & \\ 0 & \ldots\quad\ldots\quad\ldots & \\ n & \mu_1;\ldots;\mu_k;c_{k+1},\ldots;c_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[a, \beta_1, \ldots, \beta_k; c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$; $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j=1, \ldots, n$

[Extension of (6.2.14)]

$$(6.3.15)\ \mathcal{R}\left\{F\begin{bmatrix}1 & a & \\ 0 & \ldots\ \ldots\ \ldots & \\ n & \lambda_1, \nu_1 : \ldots ; \lambda_n, \nu_n & \\ 0 & \ldots\ \ldots\ \ldots & x_1 t_1, \ldots, x_n t_n \\ 0 & \ldots\ \ldots\ \ldots & \\ n & \mu_1; \ldots; \mu_k; \mu_{k+1}, c_{k+1}, \ldots, c_n, \mu_n & \end{bmatrix}\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a, \beta_1, \ldots, \beta_n; c_{k+1}, \ldots, c_n ; x_1, \ldots, x_n\right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$: $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j=1, \ldots, n$.

[Extension of (6.2.15)]

$$(6.3.16)\ \mathcal{R}\left\{F\begin{bmatrix}0 & \ldots\ \ldots\ \ldots & \\ 1 & (a;-) & \\ n & \lambda_1, \nu_1; \ldots; \lambda_k, \nu_k; a_{k+1}; \ldots; a_n & \\ 1 & c & x_1 t_1, \ldots, x_n t_n \\ 0 & \ldots\ \ldots\ \ldots & x_{k+1}, \ldots, x_n \\ k & \mu_1; \ldots; \mu_k & \end{bmatrix}\right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a, a_{k+1}, \ldots, a_n, \beta_1, \ldots, \beta_k; c; x_1, \ldots, x_n\right],$$

provided $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$; $0 < Re(\beta_j) < Re(\nu_j)$, $j=1, \ldots, k$.

[Extension of (6.2.16)]

$$(6.3.17)\ \mathcal{R}\left\{F\begin{bmatrix}1 & a & \\ 0 & \ldots\ \ldots\ \ldots & \\ n & \lambda_1, \nu_1; \ldots; \lambda_n, \nu_n & x_1 t_1, \ldots, x_n t_n \\ 0 & \ldots\ \ldots\ \ldots & \\ 1 & (c;-) & \\ n & \mu_1; \ldots; \mu_n & \end{bmatrix}\right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{D} \left[a, \beta_1, \dots, \beta_k ; c ; \; x_1, \dots, x_n \right] ,$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j = 1, \dots, n$.

[Extension of (6.2.17)]

$$(6.3.18) \quad R \left\{ F \left[\begin{array}{c|c|c} 0 & \dots \quad \dots \quad \dots & \\ 1 & (a; -) & \\ n & \lambda_1, \nu_1 ; \dots ; \lambda_n, \nu_n & x_1 t_1, \dots, x_n t_n \\ 1 & c & \\ 0 & \dots \quad \dots \quad \dots & \\ n & \mu_1 ; \dots ; \mu_n & \end{array} \right] \right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{D} \left[a, \beta_1, \dots, \beta_n ; x_1, \dots, x_n \right] ,$$

provided $0 < Re(\beta_j) < Re(\nu_j)$; $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j = 1, \dots, n$.

[Extension of (6.2.18)]

$$(6.2.19) \quad R \left\{ F \left[\begin{array}{c|c|c} 1 & b & \\ 1 & (a; -) & \\ n & \lambda_1 ; \dots ; \lambda_n & x_1 t_1, \dots, x_n t_n \\ 0 & \dots \quad \dots \quad \dots & \\ 0 & \dots \quad \dots \quad \dots & \\ n & \beta_1, \mu_1 ; \dots ; \beta_n ; \mu_n & \end{array} \right] \right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{C} \left[a, b ; \nu_1, \dots, \nu_n ; \; x_1, \dots, x_n \right] ,$$

provided $0 < Re(\beta_j) < Re(\nu_j)$; $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j = 1, \dots, n$.

[Extension of (6.2.19)]

$$(6.3.20) \quad R \left\{ F \left[\begin{array}{c|c|c} 1 & a & \\ 1 & (-; b) & \\ n & b_1 ; \dots ; b_n ; \lambda_{k+1} ; \dots ; \lambda_n & x_1, \dots, x_k, x_{k+1} t_{k+1}, \\ 0 & \dots \quad \dots \quad \dots & \dots, x_n t_n \\ 0 & \dots \quad \dots \quad \dots & \\ n-k & \beta_{k+1}, \mu_{k+1} ; \dots ; \beta_n , \mu_n & \end{array} \right] \right\}$$

$$= {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\,[a,b,b_1,\dots,b_k;\nu_{k+1},\dots,\nu_n;x_1,\dots,x_n] ,$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=k+1,\dots,n$.

[Extension of (6.2.20)]

$$(6.3.21)\quad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 0 & \dots\ \dots\ \dots & \\ n & b_1;\dots;b_k;\lambda_{k+1};\dots;\lambda_n & x_1,\dots,x_k,x_{k+1}t_{k+1}, \\ 0 & \dots\ \dots\ \dots & \\ 0 & \dots\ \dots\ \dots & \dots,\ x_nt_n \\ n-k & \beta_{k+1},\mu_{k+1};\dots;\beta_n,\mu_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\,[a,b_1,\dots,b_k;\nu_{k+1},\dots,\nu_n;x_1,\dots,x_n] ,$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=k+1,\dots,n$.

[Extension of (6.2.21)]

$$(6.3.22)\quad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 0 & \dots\ \dots\ \dots & \\ n & b_1;\dots;b_k;b_{k+1},\lambda_{k+1};\dots;b_n,\lambda_n & x_1,\dots,x_k, \\ 0 & \dots\ \dots\ \dots & x_{k+1}t_{k+1},\dots,x_nt_n \\ 0 & \dots\ \dots\ \dots & \\ n-k & \beta_{k+1},\mu_{k+1};\dots;\beta_n,\mu_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\,[a,b_1,\dots,b_n;\nu_{k+1},\dots,\nu_n;x_1,\dots,x_n] ,$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=k+1,\dots,n$.

[Extension of (6.2.22)]

$$(6.3.23)\quad R\left\{F\left[\begin{array}{c|c|c} 0 & \dots\ \dots\ \dots & \\ 1 & (a;-) & \\ n & b_1;\dots;b_k;\nu_{k+1},\lambda_{k+1};\dots;\nu_n,\lambda_n & x_1,\dots,x_k, \\ 1 & c & x_{k+1}t_{k+1},\dots,x_nt_n \\ 0 & \dots\ \dots\ \dots & \\ n-k & \mu_{k+1};\dots;\mu_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(3)}\phi^{(n)}_{BD} \left[a, \beta_{k+1}, \ldots, \beta_n, b_1, \ldots, b_k; c; \; x_1, \ldots, x_n \right] ,$$

valid if $0 < Re(\beta_j) < Re(\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j=k+1, \ldots, n$.

[Extension of (6.2.23)]

$$(6.3.24) \; R\left\{ F\left[\begin{array}{c|l|l} 2 & \nu_1 ; \lambda_1 & \\ 1 & \quad (-;b) & \\ n & b_1 ; \nu_2, \lambda_2 ; \ldots ; \nu_k, \lambda_k ; \lambda_{k+1} ; \ldots ; \lambda_n & x_1 t_1, x_1 t_1 x_2 t_2, \\ 1 & \mu_1 & \ldots, x_1 t_1 x_n t_n \\ 0 & \ldots \quad \ldots \quad \ldots & \\ n-1 & \mu_2 ; \ldots ; \mu_k ; \beta_{k+1}, \mu_{k+1} ; \ldots ; \beta_n, \mu_n & \end{array} \right] \right\}$$

$$= {}^{(k)}_{(5)}\phi^{(n)}_{CD} \left[\beta_1, b, b_1, \beta_2, \ldots, \beta_k ; \nu_{k+1}, \ldots, \nu_n ; \; x_1, x_1 x_2, \ldots, x_1 x_n \right] ,$$

valid if $0 < Re(\beta_j) < Re(\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j=1, \ldots, n$.

[Extension of (6.2.24)]

which suggests **k** results in the following unified form :

$$(6.3.25) \; R\left\{ F\left[\begin{array}{c|l|l} 2 & \nu_i ; \lambda_i & \\ 1 & \quad (-;b) & \\ n & \nu_1, \lambda_1 ; \ldots ; \nu_{i-1}, \lambda_{i-1} ; b_i ; \nu_{i+1}, \lambda_{i+1} ; & x_1 t_1 x_i t_i, \ldots, x_{i-1} t_{i-1} \cdot \\ & \quad \ldots ; \nu_k, \lambda_k ; \lambda_{k+1} ; \ldots ; \lambda_n & x_i t_i, x_i t_i, x_{i+1} t_{i+1} x_i t_i, \\ 1 & \mu_i & \ldots, x_n t_n x_i t_i \\ 0 & \ldots \quad \ldots \quad \ldots & \\ n-1 & \mu_1 ; \ldots ; \mu_{i-1} ; \mu_{i+1} ; \ldots ; \mu_k ; \mu_{k+1}, \beta_{k+1} ; & \\ & \quad \ldots ; \mu_n, \beta_n & \end{array} \right] \right\}$$

$$= {}^{(k)}_{(5)}\phi^{(n)}_{CD} \left[\beta_i, b, \beta_1, \ldots, \beta_{i-1}, b_i, \beta_{i+1}, \ldots, \beta_k ; \nu_{k+1}, \ldots, \nu_n ; x_1 x_i, \ldots, x_{i-1} x_i, \; x_i, x_{i+1} x_i, \ldots, x_n x_i \right] ,$$

valid if $0 < Re(\beta_j) < Re(\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j=1, \ldots n$.
and $i=1, \ldots, k$.

which is the Extension of (6.2.25) .

$$(6.3.26)\; R\left\{ F\begin{bmatrix} 2 & \nu_1;\lambda_1 & \\ 0 & \cdots \quad \cdots \quad \cdots & \\ n & b_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k;\lambda_{k+1};\ldots;\lambda_n & x_1t_1,\; x_1t_1x_2t_2, \\ 1 & \mu_1 & \ldots,x_1t_1x_nt_n \\ 0 & \cdots \quad \cdots \quad \cdots & \\ n-1 & \mu_2;\ldots;\mu_k;\beta_{k+1},\mu_{k+1};\ldots;\beta_n,\mu_n & \end{bmatrix}\right\}$$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[\beta_1, b_1, \beta_2,\ldots,\beta_k;\nu_{k+1},\ldots,\nu_n; x_1, x_1x_2,\ldots,x_1x_n\right],$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=1,\ldots,n$.

[Extension of (6.2.26)]

which suggests k results in the following unified form :

$$(6.3.27)\; R\left\{ F\begin{bmatrix} 2 & \nu_i;\lambda_i & \\ 0 & \cdots \quad \cdots \quad \cdots & \\ n & \nu_1,\lambda_1;\ldots;\nu_{i-1},\lambda_{i-1};b_i;\nu_{i+1},\lambda_{i+1}; & \\ & \ldots;\nu_k,\lambda_k;\lambda_{k+1};\ldots;\lambda_n & x_1t_1x_it_i,\ldots,x_{1-1}t_{i-1}, \\ 1 & \mu_i & x_it_i, x_it_i, x_{i+1}x_{i+1}x_it_i \\ 0 & \cdots \quad \cdots \quad \cdots & \\ n-1 & \mu_1;\ldots;\mu_{i-1};\mu_{i+1};\ldots;\mu_k;\mu_{k+1},\beta_{k+1}; & \ldots,x_nt_nx_it_i \\ & ;\ldots;\beta_n,\mu_n & \end{bmatrix}\right\}$$

$$= {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[\beta_i,\beta_1,\ldots,\beta_{i-1},b_i,\beta_{i+1},\ldots,\beta_k;\nu_{k+1},\ldots,\nu_n;\; x_1x_i,\ldots,x_{i-1}x_i,\; x_i, x_{i+1}x_i,\ldots, x_nx_i\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$,

$j=1,\ldots,n$ and $i=1,\ldots,k$.

[Extension of (6.2.27)]

$$(6.3.28)\quad R\left\{F\left[\begin{array}{c|l|l} 2 & \nu_1;\lambda_1 & \\ 0 & \cdots\quad\cdots\quad\cdots & \\ n & b_1;\nu_2,\lambda_2;\ \ldots;\nu_n,\lambda_n & \\ 1 & \mu_1 & x_1t_1, x_1t_1x_2t_2,\ldots, \\ 0 & \cdots\quad\cdots\quad\cdots & x_1x_nt_1t_n \\ n-1 & \mu_2;\ldots;\mu_k;\mu_{k+1},c_{k+1};\ldots;\mu_n,c_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[\beta_1, b_1, \beta_2,\ldots,\beta_n; c_{k+1},\ldots, c_n; x_1, x_1x_2,\ldots,x_1x_n\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\mu_j+\lambda_j-\beta_j)$, $j=1,\ldots,n$.

[Extension of (6.2.28)]

which suggests the n results in the following unified form :

$$(6.3.29)\quad R\left\{F\left[\begin{array}{c|l|l} 2 & \nu_i;\lambda_i & \\ 0 & \cdots\quad\cdots\quad\cdots & \\ n & \nu_1,\lambda_1;\ldots;\nu_{i-1},\lambda_{i-1};b_i;\nu_{i+1}, & \\ & \lambda_{i+1};\ldots;\nu_{k+1},\lambda_{k+1};\ldots;\nu_n,\lambda_n & x_1t_1x_it_i,\ldots,x_{i-1}t_{i-1}\cdot \\ 1 & \mu_i & x_it_i, x_it_i, x_{i+1}t_{i+1}x_it_i, \\ 0 & \cdots\quad\cdots\quad\cdots & \ldots,x_nt_nx_it_i \\ n-1 & \mu_1;\ldots;\mu_{i-1};\mu_{i+1};\ldots;\mu_k; & \\ & \mu_{k+1},c_{k+1};\ldots;\mu_n,c_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[\beta_i, \beta_1,\ldots,\beta_{i-1}, b_i, \beta_{i+1},\ldots,\beta_n; c_{k+1},\ldots,c_n; x_1x_i,\ldots,x_{i-1}x_i, x_i, x_{i+1}x_i,\ldots, x_nx_i\right],$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$,

$j=1,\ldots,n$, and $i=1,\ldots,n$.

[Extension of (6.2.29)]

$$(6.3.30)\quad R\left\{F\left[\begin{array}{c|l|l} 2 & \nu_1;\lambda_1 & \\ 0 & \cdots\quad\cdots\quad\cdots & \\ n & b_1;\lambda_2,\nu_2;\ldots;\lambda_n,\nu_n & \\ 1 & \mu_1 & x_1t_1, x_1t_1x_2t_2,\ldots,x_1t_1x_nt_n \\ 1 & (c;-) & \\ n-1 & \mu_2;\ \ldots;\mu_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi_D^{(n)}\left[\beta_1, b_1, \beta_2, \ldots, \beta_n; c; x_1, x_1x_2, \ldots, x_1x_n\right],$$

valid if $0 < Re(\beta_j) < Re(\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j=1, \ldots, n$.

[Extension of (6.2.30)]

which suggests n results in the following unified form :

$$(6.3.31)\quad R\left\{F\left[\begin{array}{c|c|c} 2 & \nu_i;\lambda_i & \\ 0 & \cdots\quad\cdots\quad\cdots & \\ n & \nu_1,\lambda_1;\ldots;\nu_{i-1},\lambda_{i-1};b_i; & x_1x_it_1t_i,\ldots,x_{i-1}t_{i-1}x_it_i, \\ & \nu_{i+1},\lambda_{i+1};\ldots;\nu_n,\lambda_n & x_it_i,x_{i+1}t_{i+1}x_it_i,\ldots, \\ 1 & \mu_i & x_nt_nx_it_i \\ 1 & (c;-) & \\ n-1 & \mu_1;\ldots;\mu_{i-1};\mu_{i+1};\ldots;\mu_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi_D^{(n)}\left[\beta_i, \beta_1, \ldots, \beta_{i-1}, b_i, \beta_{i+1}, \ldots, \beta_n; c; x_1x_i, \ldots, x_{i-1}x_i, x_i, x_{i+1}x_i, \ldots, x_nx_i\right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$,

$j=1, \ldots, n$. and $i=1, \ldots, n$. [Extension of (6.3.31)]

$$(6.3.32)\quad R\left\{F\left[\begin{array}{c|c|c} 1 & \lambda_1 & \\ 1 & (a;-) & \\ n & b_1;\lambda_2,\nu_2;\ldots;\lambda_n,\nu_n & x_1t_1, x_1t_1x_2t_2, \ldots, x_1t_1x_nt_n \\ 2 & \beta_1;\mu_1 & \\ 0 & \cdots\quad\cdots\quad\cdots & \\ n-1 & \mu_2;\ldots;\mu_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi_D^{(n)}\left[a, b_1, \beta_2, \ldots, \beta_n; \nu_1; x_1, x_1x_2, \ldots, x_1x_n\right],$$

valid if $0 < Re(\beta_j) < (\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$, $j=1, \ldots, n$.

[Extension of (6.2.32)]

which suggests n results in the following unified form :

$$(6.3.33)\quad R\left\{ F\left[\begin{array}{c|l|l} 1 & \lambda_i & \\ 1 & (a;-) & \\ n & \nu_1,\lambda_1;\ldots;\nu_{i-1},\lambda_{i-1};b_i; & x_1t_1x_it_i,\ldots,x_{i-1}t_{i-1}x_it_i, \\ & \nu_{i+1},\lambda_{i+1};\ldots;\nu_n,\lambda_n & x_it_i,x_{i+1}t_{i+1}x_it_i,\ldots, \\ 2 & \beta_i\ ;\mu_i & x_nt_nx_it_i \\ 0 & \ldots\quad\ldots\quad\ldots & \\ n-1 & \mu_1;\ldots;\mu_{i-1};\mu_{i+1},\ldots,\mu_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[a,\beta_1,\ldots,\beta_{i-1},b_i,\beta_{i+1},\ldots,\beta_n;\nu_i;x_1x_i,\ldots,x_{i-1}x_i,x_i,x_{i+1}x_i,\ldots,x_nx_i\right],$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=1,\ldots,n$. and $i=1,\ldots,n$. [Extension of (6.3.33)]

$$(6.3.34)\quad R\left\{ F\left[\begin{array}{c|l|l} 2 & \nu_1;\lambda_1 & \\ 1 & (a;-) & \\ n-1 & \lambda_2;\ldots;\lambda_n & x_1t_1,x_1t_1x_2t_2,\ldots,x_1t_1x_nt_n \\ 1 & \mu_1 & \\ 0 & \ldots\quad\ldots\quad\ldots & \\ n & c_1,\mu_2,\beta_2;\ldots;\mu_n,\beta_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[\beta_1,a;c_1,\nu_2,\ldots,\nu_n;x_1,x_1x_2,\ldots,x_1x_n\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=1,\ldots,n$.

[Extension of (6.2.34)]

which suggests n results in the following unified form :

$$(6.3.35)\quad R\left\{ F\left[\begin{array}{c|l|l} 2 & \nu_i;\lambda_i & \\ 1 & (a;-) & \\ n-1 & \lambda_1;\ldots;\lambda_{i-1};\lambda_{i+1};\ldots;\lambda_n & \\ 1 & \mu_i & x_1t_1x_it_i,\ldots,x_{i-1}t_{i-1}x_it_i, \\ 0 & \ldots\quad\ldots\quad\ldots & x_it_i,x_{i+1}t_{i+1}x_it_i,\ldots, \\ n & \beta_1,\mu_1;\ldots;\beta_{i-1};\mu_{i-1}; & \\ & c_i;\beta_{i+1},\mu_{i+1};\ldots;\beta_n,\mu_n & x_nt_nx_it_i \end{array}\right]\right\}$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{C}\left[\beta_i, a; \nu_1, \ldots, \nu_{i-1}, c_i, \nu_{i+1}, \ldots, \nu_n; x_1 x_i, \ldots, x_{i-1} x_i, x_i, x_{i+1} x_i, \ldots, x_n x_i\right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$,

$j = 1, \ldots, n$ and $i = 1, \ldots, n$ [Extension of (6.2.35)]

$$(6.3.36)\quad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 1 & (-;\nu_{k+1}, \lambda_{k+1}) & \\ n-1 & b_1;\ldots;b_k;\lambda_{k+2};\ldots;\lambda_n & x_1t_1,\ldots,x_kt_k, x_{k+1}t_{k+1}, \\ 0 & \ldots \quad \ldots \quad \ldots & x_{k+1}t_{k+1}x_{k+2}t_{k+2},\ldots, \\ 1 & (-;\mu_{k+1}) & x_nt_nx_{k+1}t_{k+1} \\ n-k & c_{k+1};\beta_{k+2},\mu_{k+2};\ldots;\beta_n,\mu_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(5)}\phi^{(n)}_{CD}\left[a, \beta_{k+1}, b_1, \ldots, b_k; c_{k+1}, \nu_{k+2}, \ldots, \nu_n; x_1, \ldots, x_k, x_{k+1}, x_{k+1}x_{k+2}, \ldots, x_{k+1}x_n\right],$$

valid if $0 < Re(\beta_j) < Re(\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$,

$j = k+1, \ldots, n$. [Extension of (6.2.26)]

which suggests n-k results in the following unified form :

$$(6.3.37)\quad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 1 & (-;\nu_i, \lambda_i) & \\ n-1 & b_1;\ldots;b_k;\lambda_{k+1};\ldots;\lambda_{i-1}; & x_1t_1,\ldots,x_kt_k, x_{k+1}x_it_{k+1}t_i \\ & \lambda_{i+1};\ldots;\lambda_n & ,\ldots, x_it_ix_{i-1}t_{i-1}, x_it_i, \\ 0 & \ldots \quad \ldots \quad \ldots & x_it_ix_{i+1}t_{i+1},\ldots,x_it_ix_nt_n \\ 1 & (-;\mu_i) & \\ n-k & \beta_{k+1};\ldots;\beta_{i-1};c_i; & \\ & \beta_{i+1};\ldots;\beta_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(5)}\phi^{(n)}_{CD}\left[a, \beta_i, b_1, \ldots, b_k; \nu_{k+1}, \ldots, \nu_{i-1}, c_i, \nu_{i+1}, \ldots, \nu_n; x_1, \ldots, x_k, x_ix_{k+1}, \ldots, x_ix_{i-1}, x_i, x_ix_{i+1}, \ldots, x_nx_i\right],$$

provided $\quad 0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)\quad,\quad 0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)\quad,$

$j=k+1,\dots,n \quad,\quad i=k+1,\dots,n\quad.$

[Extension of (6.2.37)]

$$(6.3.38)\quad R\left\{F\left[\begin{array}{c|c|c} 0 & \dots\quad\dots\quad\dots & \\ 1 & (\lambda_1,\nu_1;-) & \\ n & b_1;\lambda_2,\nu_2;\dots;\lambda_k,\nu_k;a_{k+1};\dots;a_n & x_1t_1,x_1t_1x_2t_2,\dots, \\ 1 & c & x_1t_1x_kt_k,x_{k+1}t_{k+1}, \\ 1 & (\mu_1;-) & \dots,x_nt_n \\ k-1 & \mu_2;\dots;\mu_k & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[\beta_1,a_{k+1},\dots,a_n,b_1,\beta_2,\dots,\beta_k;c;x_1,x_1x_2,\dots,x_1x_k,x_{k+1},\dots,x_n\right],$$

valid if $\quad 0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)\quad,\quad 0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)\quad,\ j=1,\dots,k\ .$

[Extension of (6.2.38)]

which suggests k results in the following unified form :

$$(6.3.39)\quad R\left\{F\left[\begin{array}{c|c|c} 0 & \dots\quad\dots\quad\dots & \\ 1 & (\lambda_i,\nu_i;-) & \\ n & \lambda_1,\nu_1;\dots;\lambda_{i-1},\nu_{i-1};b_i;\lambda_{i+1}, & x_1x_it_1t_i,\dots,x_{i-1}t_{i-1} \\ & \nu_{i+1};\dots;\lambda_k,\nu_k;a_{k+1};\dots;a_n & x_it_i,x_it_i,x_{i+1}t_{i+1}x_it_i \\ 1 & c & \dots,x_kt_kx_it_i,x_{k+1}t_{k+1}, \\ 1 & (\mu_i;-) & \dots,x_nt_n \\ k-1 & \mu_1;\mu_2;\dots;\mu_{i-1};\mu_{i+1};\dots;\mu_k & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[\beta_i,a_{k+1},\dots,a_n,\beta_1,\dots,\beta_{i-1},b_i,\beta_{i+1},\dots,\beta_k;c;\ x_1x_i,\dots,x_{i-1}x_i,\ x_i,x_{i+1}x_i,x_{k+1},\dots,x_n\right],$$

valid if $\quad 0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)\quad,\quad 0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)\quad;$

$j=1,\dots,k$ and $\ i=1,\dots,k\ .$

[Extension of (6.2.39)]

$$(6.3.40)\quad R\left\{F\left[\begin{array}{c|l|l} 1 & a & x_1t_1,x_1t_1x_2t_2, \\ 1 & (\lambda_1;-) & \ldots,x_1t_1x_kt_k, \\ n & b_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k;b_{k+1};\ldots;b_n & x_{k+1}t_{k+1},\ldots,x_nt_n \\ 0 & \ldots\quad\ldots\quad\ldots & \\ 1 & (\beta_1,\mu_1;-) & \\ k-1 & \mu_2;\ldots;\mu_k & \end{array}\right]\right.$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a,b_1,\beta_2,\ldots,\beta_k,b_{k+1},\ldots,b_n;\nu_1;x_1x_2,\ldots,x_1x_k,x_{k+1},\ldots,x_n\right],$$

provided $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=1,\ldots,k$.

which suggests k results in the following unified form :

$$(6.3.41)\quad R\left\{F\left[\begin{array}{c|l|l} 1 & a & x_1t_1x_it_i,\ldots, \\ 1 & (\lambda_i;-) & x_{i-1}t_{i-1}x_it_i,x_it_i \\ n & \nu_1,\lambda_1;\ldots;\nu_{i-1},\lambda_{i-1};b_i;\nu_{i+1},\lambda_{i+1} & x_{i+1}t_{i+1}x_it_i,\ldots, \\ & \ldots;\nu_k,\lambda_k;b_{k+1};\ldots;b_n & x_kt_kx_it_i,x_{k+1}t_{k+1}, \\ 0 & \ldots\quad\ldots\quad\ldots & \ldots,\ x_nt_n \\ 1 & (\mu_i,\beta_i;-) & \\ k-1 & \mu_1;\ldots;\mu_{i-1};\mu_{i+1};\ldots;\mu_k & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[a,\beta_1,\ldots,\beta_{i-1},b_i,\beta_{i+1},\ldots,\beta_k,b_{k+1},\ldots,b_n;\nu_i,\ x_1x_i,\ldots,x_{i-1}x_i,\right.$$

$$\left.x_i,x_{i+1}x_i,\ldots,x_kx_i,x_{k+1},\ldots,x_n\right],$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$,

$j=1,\ldots,k$ and $i=1,\ldots,k$.

[Extension of (6.2.41)]

$$(6.3.42)\quad R\left\{F\left[\begin{array}{c|l|l} 0 & \ldots\quad\ldots\quad\ldots & \\ 1 & (\nu_1,\lambda_1;-) & \\ n & b_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k;b_{k+1};\ldots;b_n & x_1t_1,x_1t_1x_2t_2,\ldots, \\ 1 & c & x_it_ix_kt_k,x_{k+1}t_{k+1}, \\ 1 & (\mu_1;-) & \ldots,x_nt_n \\ k-1 & \mu_2;\ldots;\mu_k & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[\beta_1, b_1, \beta_2, \ldots, \beta_k, b_{k+1}, \ldots, b_n; c; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots x_n\right],$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\rho_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=1,\ldots,k$.

[Extension of (6.2.42)]

which suggests k results in the following unified form :

$$(6.3.43)\quad R\left\{F\left[\begin{array}{c|l|l} 0 & \ldots\quad \ldots\quad \ldots & x_1t_1x_it_i,\ldots, \\ 1 & (\nu_i,\lambda_i;-) & x_{i-1}t_{i-1}x_it_i, x_it_i \\ n & \lambda_1,\nu_1;\ldots;\lambda_{i-1},\nu_{i-1};b_i;\lambda_{i+1},\nu_{i+1} & x_{i+1}t_{i+1}x_it_i, \\ & ;\ldots;\lambda_k,\nu_k;b_{k+1},\ldots;b_n & \ldots,x_kt_kx_it_i, \\ 1 & c & x_{k+1}t_{k+1},\ldots, \\ 1 & (\mu_i;-) & x_nt_n \\ k-1 & \mu_1;\ldots;\mu_{i-1};\mu_{i+1};\ldots;\mu_k & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[\beta_i, \beta_1, \ldots, \beta_{i-1}, b_i, \beta_{i+1}, \ldots, \beta_k, b_{k+1}, \ldots, b_n; c; x_1x_i, \ldots, x_{i-1}x_i, x_i, x_{i+1}x_i, \ldots, x_kx_i, x_{k+1}, \ldots, x_n\right],$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$,

$j=1,\ldots,k$ and $i=1,\ldots,k$. [Extension of (6.2.43)]

$$(6.3.44)\quad R\left\{F\left[\begin{array}{c|l|l} 1 & b & \\ 1 & (\nu_1,\lambda_1;-) & \\ k-1 & \lambda_2;\ldots;\lambda_k & x_1t_1, x_1t_1x_2t_2, \\ 0 & \ldots\quad \ldots\quad \ldots & \ldots, x_1t_1x_kt_k, \\ 1 & (\mu_1;-) & x_{k+1}t_{k+1},\ldots, \\ n & c_1;\beta_2,\mu_2;\ldots;\beta_k,\mu_k;c_{k+1};\ldots;c_n & x_nt_n \end{array}\right]\right\}$$

$$= {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[b, \beta_1; c_1, \nu_2, \ldots, \nu_k, c_{k+1}, \ldots, c_n; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots, x_n\right],$$

valid if $0<\mathrm{Re}(\beta_j)<\mathrm{Re}(\nu_j)$, $0<\mathrm{Re}(\mu_j)<\mathrm{Re}(\nu_j+\lambda_j-\beta_j)$, $j=1,\ldots,k$.

[Extension of (6.2.44)]

$$(6.3.45)\quad R\left\{ F\left[\begin{array}{c|c|l} 1 & b & x_1 x_i t_1 t_i, \ldots, \\ 1 & (\nu_i, \lambda_i; -) & x_{i-1} t_{i-1} x_i t_i, x_i t_i, \\ k-1 & \lambda_1; \ldots; \lambda_{i-1}; \lambda_{i+1}; \ldots; \lambda_k & x_{i+1} t_{i+1} x_i t_i, \ldots, \\ 0 & \ldots \quad \ldots \quad \ldots & x_k t_k x_i t_i, x_{k+1} t_{k+1}, \\ 1 & (\mu_i; -) & \ldots, x_n t_n \\ n & \beta_1, \mu_1; \ldots; \beta_{i-1}, \mu_{i-1}; c_i; \beta_{i+1}, & \\ & \mu_{i+1}; \ldots; \beta_k, \mu_k; c_{k+1}; \ldots; c_n & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(1)}I^{(n)}_{C}\left[b, \beta_i; \nu_1, \ldots, \nu_{i-1}, c_i, \nu_{i+1}, \ldots, \nu_k, c_{k+1}, \ldots, c_n; x_1 x_i, \ldots, x_{i-1} x_i, x_i, x_{i+1} x_i, \ldots, x_k x_i, x_{k+1}, \ldots, x_n \right],$$

provided $0 < Re(\beta_j) < Re(\nu_j)$, $0 < Re(\mu_j) < Re(\nu_j + \lambda_j - \beta_j)$,

$j = 1, \ldots, k$ and $i = 1, \ldots, k$.

[Extension of (6.2.45)]

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FRACTIONAL DERIVATIVES OF THE MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

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FRACTIONAL DERIVATIVES OF THE MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

7.1 **Introduction** The theory and applications of fractional calculus are based largely upon the familiar differintegral operator ${}_{\alpha}D_x^{\mu}$ defined by (cf., e.g. , Oldham and Spanier [15, p. 49] ,Lavoie et. al. [11] , and Ross [16] ; see also Srivastava and Owa [30],p.356)

$$(7.1.1)\qquad {}_{\alpha}D_x^{\mu}\{f(x)\} = \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_{\alpha}^{x}(x-t)^{-\mu-1}f(t)\,dt & (\mathrm{Re}(\mu)<0), \\[2ex] \dfrac{d^m}{dx^m}\,{}_{\alpha}D_x^{\mu-m}\{f(x)\} & (0\leq \mathrm{Re}(\mu)<m;\ m\in N_0), \end{cases}$$

where $N_0 = \{0,1,2,\ldots\}$

For $\alpha=0$, equation (7.1.1) defines the classical Riemann - Liouville fractional derivative(or integral) of order μ(or $-\mu$). On the other hand when $\alpha\to\infty$, equation (7.1.1) may be identified with the definition of the familiar Weyl fractional derivative (or integral) of order μ(or $-\mu$) (see for details, Erdélyi et al. [5, chapter 13] and Samko et al. [17]). For the sake of simplicity, the special case of the fractional calculus operator ${}_{\alpha}D_x^{\mu}$ when $\alpha=0$ will be written D_x^{μ}. Thus we have

$$(7.1.2)\qquad D_x^{\mu} = {}_0D_x^{\mu}\qquad (\mu\in C)\ .$$

From this chapter a paper entitled "Fractional derivatives of confluent hypergeometric forms of Karlsson's multiple hypergeometric function ${}^{(k)}F_{CD}^{(n)}$" has been published in Pure Appl. Math. Sci. , 35(1992), 31 - 39 .

The computation of fractional derivatives (and fractional integrals) of special functions of one and more variables is important from the point of view of the usefulness of these results in (for example) the evaluation of series and integrals (cf. , e.g. , Nishimoto [13] and Srivastava [32]) , the derivation of generating functions (Srivastava and Manocha [28, chapter 5]) , and the solutions of differential and integral equations (cf.Nishimoto [13] , and Srivastava and Buschman [31, chapter 3] ; and see also Mcbride and Roach [12] , Nishimoto [14] ,and Srivastava and Saigo [29]) . Motivated by these and other avenues of applications , a number of workers have made use of the fractional calculus operator D_x^{μ} in the theory of special functions of one and more variables .

Recently, Srivastava and Goyal [19] , have derived several fractional derivative formulae involving the multivariable H- function defined by Srivastava and Panda [20, p. 271, eq.(4.1) et. seq.] and studied systematically by them (see [21 - 24] ; (see also [19]) . Some obvious special cases of the results of Srivastava and Goyal [19] were proved subsequently by Chouksey and Sharma [4]. Sharma and Singh [33], on the other hand, have recently considered some straight forward variations of the results of Srivastava and Goyal [19].

For special interest, in chapter III we have derived fractional derivatives involving hypergeometric functions of four variables.

In the present chapter, we shall derive fractional derivatives involving generalized multiple hypergeometric function of Srivastava and Daoust [18] specially under those conditions which were restricted by Srivastava and Goyal [27] , we shall also discuss their special cases to derive the fractional derivatives involving the multiple hypergeometric functions of several variables defined by Lauricella [10], Exton [6.7] , Chandel [1] , Chandel - Gupta [2] and Karlsson [9]. we shall also derive the results for confluent forms of the above multiple hypergeometric functions. Finally , we shall also derive multidimensional fractional derivatives involving multiple hypergeometric functions of several variables (see also Chandel and Vishwakarma [3]) .

7.2. FRACTIONAL DERIVATIVES

Srivastava and Goyal [27] evaluated

$$D_x^{\mu}\left\{x^k(x^{\nu}+\xi) \quad H\left[z_1 x^{\rho_1}(x+\xi)^{\sigma_1}, \ldots, z_r x^{\rho_r}(x+\xi)^{\sigma_r}\right]\right\}$$

where $H[z_1,\ldots,z_r]$ is multivariable H-function of Srivastava and Panda [20] under certain conditions including $\min(\rho_i,\sigma_i)>0$, $i=1,\ldots,r$.

Here we shall evaluate

$$D_x^{\mu}\left\{x^k(x^{\nu}+\xi)^{\lambda} F\left[z_1 x^{\rho_1}(x^{\nu}+\xi)^{-\sigma_1}, \ldots, z_r x^{\rho_r}\cdot(x^{\nu}+\xi)^{-\sigma_r}\right]\right.$$

where $\min(\nu,\rho_i,\sigma_i)>0$, and $F[z_1,\ldots,z_r]$ is generalized hypergeometric function of Srivastava and Daoust [18] defined by

$$F[z_1,\ldots,z_r] = F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1,\ldots,z_r\right),$$

which is absolutely convergent if for all x_i

$$1 + \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0 \quad , \quad i=1,\ldots,r \ .$$

The fundamental difference of both the results is that Srivastava and Goyal [27] have specially taken positive powers of $(x + \xi)$ while here we specially consider negative powers of $(x^{\nu} + \xi)$ otherwise validity of the results of both works will be destroyed.

our main results are

$$(7.2.1) \quad D_x^{\mu} \left\{ x^k (x^{\nu} + \xi)^{\lambda} \cdot F[z_1 x^{\rho_1} (x^{\nu} + \xi)^{-\sigma_1}, \ldots, z_r x^{\rho_r} (x^{\nu} + \xi)^{-\sigma_r}] \right\}$$

$$= \frac{\Gamma(1+k)}{\Gamma(1+k-\mu)} \xi^{\lambda} \cdot x^{k-\mu} \; F^{A+2:B';\ldots;B^{(r)},0}_{C+2:D';\ldots;D^{(r)},0} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)},0], \\ [(c):\Psi',\ldots,\Psi^{(r)},0], \end{matrix} \right.$$

$$[-\lambda:\sigma_1,\ldots,\sigma_r,1], [1+k:\rho_1,\ldots,\rho_r,\nu]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];-;$$

$$[-\lambda:\sigma_1,\ldots,\sigma_r,0],[1+k-\mu:\rho_1,\ldots,\rho_r,\nu]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];-;$$

$$\left. \frac{z_1 x^{\rho_1}}{\xi^{\sigma_1}}, \ldots, \frac{z_r x^{\rho_r}}{\xi^{\sigma_r}}, \frac{-x^{\nu}}{\xi} \right)$$

$Re(k-\mu) > -1$, $\min(\nu, \rho_i, \sigma_i) \geq 0$, $i=1,\ldots,r$.

and $$1 + \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0 \ .$$

$$(7.2.2) \quad D_x^{\mu}\, D_y^{\mu'} \{ x^k\, y^{k'}\, (x^l + \xi)^{\lambda}\, (y^{l'} + \eta)^{\lambda'} . F[z_1 x^{\rho_1} y^{\lambda_1} (x^l+\xi)^{-\sigma_1} . (y^{l'}+\eta)^{-\nu_1}, \ldots, z_r\, x^{\rho_r}\, y^{\lambda_r} (x^l+\xi)^{-\sigma_r} . (y^{l'}+\eta)^{-\nu_r}] \}$$

$$= \frac{\xi^{\lambda} . \eta^{\lambda'}\, x^{k-\mu}\, y^{k'-\mu'} . \Gamma(1+k)\, \Gamma(1+k')}{\Gamma(1+k-\mu)\, \Gamma(1+k'-\mu')}\; F^{A+4:B';\ldots;B^{(r)};\,0;0}_{C+4:D';\ldots;D^{(r)};\,0;0} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)},0,0], [1+k:\rho_1,\ldots,\rho_r,l,0], [1+k':\lambda_1,\ldots,\lambda_r,0,l'], \\ [(c):\Psi',\ldots,\Psi^{(r)},0,0], [1+k-\mu:\rho_1,\ldots,\rho_r,1,0], [1+k'-\mu':\lambda_1,\ldots,\lambda_r,0,l'], \end{matrix} \right.$$

$$\left. \begin{matrix} [-\lambda:\sigma_1,\ldots,\sigma_r,1,0], [-\lambda':\nu_1,\ldots,\nu_r,0,1]: [(b'):\phi'];\ldots; [(b^{(r)}):\phi^{(r)}];-;-; \\ [-\lambda:\sigma_1,\ldots,\sigma_r,0,0], [-\lambda':\nu_1,\ldots,\nu_r,0,0]:[(d'):\delta'];\ldots; [(d^{(r)}):\delta^{(r)}];-;-; \end{matrix} \; \frac{z_1\, x^{\rho_1}\, y^{\lambda_1}}{\xi^{\sigma_1}\, \eta^{\nu_1}}, \ldots, \frac{z_r\, x^{\rho_r}\, y^{\lambda_r}}{\xi^{\sigma_r}\, \eta^{\nu_r}}, \frac{-x^l}{\xi}, \frac{-y^{l'}}{\eta} \right),$$

$Re(k-\mu) > -1$, $Re(k'-\mu') > -1$, x,y are independent

$\min(l, l', \rho_i, \sigma_i, \lambda_i, \nu_i) > 0$ and

$$1 + \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0,$$

$i = 1, \ldots, r$.

<u>Proof of (7.2.1)</u> In the left hand side first, we expand multiple hypergeometric function and collect the powers of $(x^{\nu} + \xi)$, then by

binomial expansion, we collect the powers of x and finally we apply the formula [15, p.67]

$$(7.2.3) \qquad D_x^{\lambda}\left\{ x^{\lambda}\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} x^{\lambda-\mu} \quad , \operatorname{Re}(\lambda) > -1 \quad .$$

and rearrange the series to get (7.2.1) .

<u>Proof of (7.2.2)</u> Here x and y are independent therefore by making same techniues of (7.2.1) separately w. r. t. x and y , we can derive (7.2.2) .

7.3. SPECIAL CASES OF (7.2.1)

<u>Use of one fractional derivative operator</u>. In this section, we derive the following relations as the special cases of (7.2.1):

For $\lambda=0$, $\sigma_1=\ldots=\sigma_r=0$, $\nu=\rho_1,\ldots,\rho_r=1$, replacing μ by $\lambda-\mu$ and k by $\lambda-1$, (7.2.1) reduces to

$$(7.3.1) \qquad D_x^{\lambda-\mu}\left\{ x^{\lambda-1} \cdot F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1x,\ldots,z_rx\right)\right.$$

$$= \frac{\Gamma(\lambda)\, x^{\mu-1}}{\Gamma(\mu)} \cdot F^{A+1:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],[\lambda:1,\ldots,1]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}],[\mu:1,\ldots,1]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1x,\ldots,z_nx\right)$$

$Re(\lambda) > 0$ and series involved is convergent .

(7.3.2) $D_x^{\lambda-\mu}\{x^{\lambda-1} F_A^{(n)}[\mu, b_1, \ldots, b_n; c_1, \ldots, c_n; z_1 x, \ldots, z_n x]\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} F_A^{(n)}[\lambda, b_1, \ldots, b_n; c_1, \ldots, c_n; z_1 x, \ldots, z_n x],$$

$Re(\lambda) > 0$, $|z_1 x| + \ldots + |z_n x| < 1$.

(7.3.3) $D_x^{\lambda-\mu}\{x^{\lambda-1} F_B^{(n)}[a_1, \ldots, a_n, b_1, \ldots, b_n; \lambda; z_1 x, \ldots, z_n x]\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} . F_B^{(n)}[a_1, \ldots, a_n, b_1, \ldots, b_n; \mu; z_1 x, \ldots, z_n x],$$

$R(\lambda) > 0$, $\max(|z_1 x|, \ldots, |z_n x|) < 1$.

(7.3.4) $D_x^{\lambda-\mu}\{x^{\lambda-1} F_C^{(n)}[\mu, b; c_1, \ldots, c_n; xz_1, \ldots, xz_n]\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} . x^{\mu-1} F_C^{(n)}[\lambda, b; c_1, \ldots, c_n; z_1 x, \ldots, z_n x],$$

$Re(\lambda) > 0$, $|z_1 x|^{\frac{1}{2}} + \ldots + |z_n x|^{\frac{1}{2}} < 1$.

(7.3.5) $D_x^{\lambda-\mu}\{x^{\lambda-1} F_D^{(n)}[\mu, b_1, \ldots, b_n; c; z_1 x, \ldots, z_n x]\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} F_D^{(n)}[\lambda, b_1, \ldots, b_n; c; z_1 x, \ldots, z_n x],$$

$Re(\lambda) > 0$, $\max(|z_1 x|, \ldots, |z_n x|) < 1$.

Particularly for $c=\mu$, (7.3.5) reduces to the result due to Srivastava and Goyal [27, (7.3.6)] .

$$(7.3.6)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\, F_D^{(n)}[a,b_1,\ldots,b_n;\lambda;z_1x,\ldots,z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\, x^{\mu-1}\, F_D^{(n)}[a,b_1,\ldots,b_n;\mu;z_1x,\ldots,z_nx] ,$$

$$\mathrm{Re}(\lambda)>0 \quad , \quad \max(|z_1x|,\ldots,|z_nx|)<1 .$$

For $a=\lambda$, (7.3.6) reduces to the result due to Srivastava and Goyal [27,(7.3.6)]:

$$(7.3.7)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\prod_{i=1}^{n}(1-z_ix)^{-b_i}\right\} = x^{\mu-1}\frac{\Gamma(\lambda)}{\Gamma(\mu)}\, F_D^{(n)}[\lambda,b_1,\ldots,b_n; z_1x,\ldots,z_nx] ,$$

$\mathrm{Re}(\lambda)>0$; $\max\{(|z_1x|,\ldots,|z_nx|)\}<1$.
which for $n=2$, was given by earlier by Lavoie et al. [11,p.260].

On the other hand (7.3.5) when $c=\lambda$ or (7.3.6) when $a=\mu$, would similarly yield the following companion of (7.3.7) :

$$(7.3.8)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\, F_D^{(n)}[\mu,b_1,\ldots,b_n;\lambda;z_1x,\ldots,z_nx]\right\}$$

$$= x^{\mu-1}\frac{\Gamma(\lambda)}{\Gamma(\mu)}\cdot\prod_{i=1}^{n}(1-z_ix)^{-b_i} \quad , \quad \mathrm{Re}(\lambda)>0 ,$$

$$\max\{|z_1x|,\ldots,|z_nx|\}<1,$$

which does not seem to have been recorded earlier .

Here $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ are Lauricella's multiple hypergeometric function [10] .

$$(7.3.9)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\, \Phi_2^{(n)}[b_1,\ldots,b_n;\lambda;z_1x,\ldots,z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\, x^{\mu-1}\cdot \Phi_2^{(n)}[b_1,\ldots,b_n;\mu;z_1x,\ldots,z_nx] , \quad \mathrm{Re}(\lambda)>0 .$$

$$(7.3.10)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ \Psi_2^{(n)}[\mu; c_1,\ldots,c_n; z_1x,\ldots,z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ \Psi_2^{(n)}[\lambda; c_1,\ldots,c_n; z_1x,\ldots,z_nx]\ ,\quad \mathrm{Re}(\lambda)>0\ .$$

$$(7.3.11)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ \Phi_D^{(n)}[\mu, b_1,\ldots,b_{n-1}; c; z_1x,\ldots,z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ \Phi_D^{(n)}[\lambda, b_1,\ldots,b_{n-1}; c; z_1x,\ldots,z_nx]\ ,\quad \mathrm{Re}(\lambda)>0$$

$$(7.3.12)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ \Xi_1^{(n)}[a_1,\ldots,a_n,b_1,\ldots,b_{n-1};\lambda; z_1x,\ldots,z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ \Xi_1^{(n)}[a_1,\ldots,a_n,b_1,\ldots,b_{n-1};\mu; z_1x,\ldots,z_nx]\ ,$$

$\mathrm{Re}(\lambda) > 0$.

$$(7.3.13)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ \Phi_3^{(n)}[b_1,\ldots,b_{n-1};\lambda; z_1x,\ldots,z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ \Phi_3^{(n)}[b_1,\ldots,b_{n-1};\mu; z_1x,\ldots,z_nx]\ ,\quad \mathrm{Re}(\lambda)>0\ .$$

where $\Phi_2^{(n)}$, $\Psi_2^{(n)}$, $\Phi_D^{(n)}$, $\Xi_1^{(n)}$ and $\Phi_3^{(n)}$ are confluent forms of Lauricella's multiple hypergeometric functions [10] .

$$(7.3.14)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ {}^{(k)}_{(1)}E_D^{(n)}[\mu, b_1,\ldots,b_n; c, c'; z_1x,\ldots,z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ {}^{(k)}_{(1)}E_D^{(n)}[\lambda, b_1,\ldots,b_n; c, c'; z_1x,\ldots,z_nx]\ ,\quad \mathrm{Re}(\lambda)>0$$

$r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_n$, $r_k + r_n = 1$, where $|z_i x| < r_i$, $i=1,\ldots,n$.

$$(7.3.15)\quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ {}^{(k)}_{(2)}E_D^{(n)}[a, a', b_1,\ldots,b_n; \lambda; z_1x,\ldots,z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \, {}^{(k)}_{(2)}E^{(n)}_{D} \left[a, a', b_1, \ldots, b_n; \mu; z_1 x, \ldots, z_n x \right],$$

$Re(\lambda) > 0$, $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_n$, $r_k \cdot r_n = r_k + r_n$,

$|z_i x| < r_i$, $i = 1, \ldots, n$.

(7.3.16) $$D_x^{\lambda-\mu} \left\{ x^{\lambda-1} \, {}^{(k)}_{(1)}E^{(n)}_{C} \left[a, a', \mu; c_1, \ldots, c_n; z_1 x, \ldots, z_n x \right] \right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} . x^{\mu-1} \, {}^{(k)}_{(1)}E^{(n)}_{C} \left[a, a', \lambda; c_1, \ldots, c_n; z_1 x, \ldots, z_n x \right],$$

$Re(\lambda) > 0$, $(\sqrt{r_1} + \ldots + \sqrt{r_k})^2 + (\sqrt{r_{k+1}} + \ldots + \sqrt{r_n})^2 = 1$,

$|z_i x| < r_i$, $i = 1, \ldots, n$.

Here ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$ are Exton's multiple hypergeometric functions [6,7] related to Lauricella's $F^{(n)}_{D}$, while ${}^{(k)}_{(1)}E^{(n)}_{C}$ is Chandel's multiple hypergeometric function [1] related to Lauricella's $F^{(n)}_{C}$.

(7.3.17) $$D_x^{\lambda-\mu} \left\{ x^{\lambda-1} \, {}^{(k)}F^{(n)}_{AC} \left[\mu, b, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; z_1 x, \ldots, z_n x \right] \right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} . \, {}^{(k)}F^{(n)}_{AC} \left[\lambda, b, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; z_1 x, \ldots, z_n x \right],$$

$Re(\lambda) > 0$, $(|z_1 x|^{\frac{1}{2}} + \ldots + |z_k x|^{\frac{1}{2}})^2 + |z_{k+1} x| + \ldots + |z_n x| < 1$,

(7.3.18) $$D_x^{\lambda-\mu} \left\{ x^{\lambda-1} \, {}^{(k)}F^{(n)}_{BD} \left[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n; \lambda; z_1 x, \ldots, z_n x \right] \right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \, {}^{(k)}F^{(n)}_{BD} \left[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n; \mu; z_1 x, \ldots, z_n x \right], \; Re(\lambda) > 0 .$$

$\max(|z_1 x|, \ldots, |z_n x|) < 1$.

(7.3.19) $$D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ {}^{(k)}F_{AD}^{(n)}[\mu, b_1, \dots, b_n; c, c_{k+1}, \dots, c_n; z_1x, \dots, z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1}\ {}^{(k)}F_{AD}^{(n)}[\lambda, b_1, \dots, b_n; c, c_{k+1}, \dots, c_n; z_1x, \dots, z_nx], \quad Re(\lambda) > 0 .$$

$\max(|z_1x|, \dots, |z_kx|) + |z_{k+1}x| + \dots + |z_nx| < 1$.

(7.3.20) $$D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ {}^{(k)}_{(1)}\Phi_{AC}^{(n)}[\mu, b; c_1, \dots, c_n; z_1x, \dots, z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1}\ {}^{(k)}_{(1)}\Phi_{AC}^{(n)}[\lambda, b; c_1, \dots, c_n; z_1x, \dots, z_nx], \quad Re(\lambda) > 0 .$$

(7.3.21) $$D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ {}^{(k)}_{(2)}\Phi_{AC}^{(n)}[\mu, b_{k+1}, \dots, b_n; c_1, \dots, c_n; z_1x, \dots, z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1}\ {}^{(k)}_{(2)}\Phi_{AC}^{(n)}[\lambda, b_{k+1}, \dots, b_n; c_1, \dots, c_n; z_1x, \dots, z_nx], \quad Re(\lambda) > 0 .$$

(7.3.22) $$D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ {}^{(k)}_{(1)}\Phi_{AD}^{(n)}[\mu, b_1, \dots, b_n; c; z_1x, \dots, z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1}\ {}^{(k)}_{(1)}\Phi_{AD}^{(n)}[\lambda, b_1, \dots, b_n; c; z_1x, \dots, z_nx], \quad Re(\lambda) > 0 .$$

(7.3.23) $$D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ {}^{(k)}_{(1)}\Phi_{BD}^{(n)}[a, b_1, \dots, b_n; \lambda; z_1x, \dots, z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1}\ {}^{(k)}_{(1)}\Phi_{BD}^{(n)}[a, b_1, \dots, b_n; \mu; z_1x, \dots, z_nx], \quad Re(\lambda) > 0 .$$

(7.3.24) $$D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\ {}^{(k)}_{(2)}\Phi_{BD}^{(n)}[a_{k+1}, \dots, a_n, b_1, \dots, b_n; \lambda; z_1x, \dots, z_nx]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1}\ {}^{(k)}_{(2)}\Phi_{BD}^{(n)}[a_{k+1}, \dots, a_n, b_1, \dots, b_n; \mu; z_1x, \dots, z_nx], \quad Re(\lambda) > 0 .$$

Here ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$, ${}^{(k)}F_{BD}^{(n)}$ are Lauricella's intermediate function

while ${}^{(k)}_{(1)}\Phi^{(n)}_{AC}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{AC}$, ${}^{(k)}_{(1)}\Phi^{(n)}_{AD}$, ${}^{(k)}_{(1)}\Phi^{(n)}_{BD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{BD}$ are their confluent forms introduced by Chandel and Gupta [2] .

(7.3.25) $D_x^{\lambda-\mu}\left\{ x^{\lambda-1}\; {}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\ldots,b_n;\lambda,c';z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n]\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\, x^{\mu-1}\; {}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\ldots,b_n;\mu,c';z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n] ,$$

$Re(\lambda)>0$, $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_{k+1}+r_n=1$,

$|z_i|<r_i$, $i=1,\ldots,k$, $|z_ix|<r_i$, $i=k+1,\ldots,n$.

(7.3.26) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}\; {}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\ldots,b_n;c,\lambda;z_1,\ldots,z_k,z_{k+1}x,\ldots,z_nx]\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\, x^{\mu-1}\; {}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\ldots,b_n;c,\mu;z_1,\ldots,z_k,z_{k+1}x,\ldots,z_n] ,$$

$Re(\lambda)>0$, $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_{k+1}+r_n=1$,

$|z_i|<r_i$, $i=1,\ldots,k$. $|z_ix|<r_i$, $i=k+1,\ldots,n$.

(7.3.27) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}[\mu,a',b_1,\ldots,b_n;c;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n]\right.$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\, x^{\mu-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}[\lambda,a',b_1,\ldots,b_n;c;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n] ,$$

$Re(\lambda)>0$, $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_k.r_n=r_k+r_n$,

$|z_ix|<r_i$, $i=1,\ldots,k$, $|z_i|<r_i$, $i=k+1,\ldots,n$.

(7.3.28) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}[a,\mu,b_1,\ldots,b_n;c;z_1,\ldots,z_k,z_{k+1}x,\ldots,xz_n]\right.$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\, x^{\mu-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}[a,\lambda,b_1,\ldots,b_n;c;z_1,\ldots,z_k,z_{k+1}x,\ldots,z_nx],$$

$Re(\lambda)>0$, $r_1=\dots=r_k$, $r_{k+1}=\dots=r_n$, $r_k \cdot r_n = r_k + r_n$,

$|z_i|<r_i$, $i=1,\dots,k$, $|z_i x|<r_i$, $i=k+1,\dots,n$.

(7.3.29) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}_{(1)}E_C^{(1)}[\mu, a', b; c_1,\dots,c_n; z_1x,\dots,z_kx, z_{k+1},\dots,z_n]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ {}^{(k)}_{(1)}E_C^{(n)}[\lambda, a', b; c_1,\dots,c_n; z_1x,\dots,z_kx, z_{k+1},\dots,z_n] ,$$

$Re(\lambda)>0$, $(\sqrt{r_1}+\dots+\sqrt{r_k})^2 + (\sqrt{r_{k+1}}+\dots+\sqrt{r_n})^2 = 1$,

$|z_i x|<r_i$, $i=1,\dots,k$, $|z_i|<r_i$, $i=k+1,\dots,n$.

(7.3.30) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}_{(1)}E_C^{(n)}[a,\mu,b; c_1,\dots,c_n; z_1,\dots,z_k, z_{k+1}x,\dots,z_nx]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\cdot\ {}^{(k)}_{(1)}E_C^{(n)}[a,\lambda,b; c_1,\dots,c_n; z_1,\dots,z_k, z_{k+1}x,\dots,z_nx] ,$$

$Re(\lambda)>0$, $(\sqrt{r_1}+\dots+\sqrt{r_k})^2 + (\sqrt{r_{k+1}}+\dots+\sqrt{r_n})^2 = 1$,

$|z_i|<r_i$, $i=1,\dots,k$. $|z_i x|<r_i$, $i=k+1,\dots,n$.

(7.3.31) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}F_{AC}^{(n)}[a,\mu,b_{k+1},\dots,b_n; c_1,\dots,c_n; z_1x,\dots,z_kx, z_{k+1},\dots,z_n]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\cdot\ {}^{(k)}F_{AC}^{(n)}[a,\lambda,b_{k+1},\dots,b_n; c_1,\dots,c_n; z_1x,\dots,z_kx, z_{k+1},\dots,z_n] ,$$

$Re(\lambda)>0$, $(|z_1x|^{\frac12}+\dots+|z_kx|^{\frac12})^2 + |z_{k+1}| + \dots + |z_n| < 1$,

(7.3.32) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}F_{AC}^{(n)}[a,\mu,b_{k+1},\dots,b_n; c_1,\dots,c_n; z_1x,\dots,z_kx, z_{k+1},\dots,z_n]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\cdot\ {}^{(k)}F_{AC}^{(n)}[a,\lambda,b_{k+1},\dots,b_n; c_1,\dots,c_n; z_1x,\dots,z_kx, z_{k+1},\dots,z_n],$$

$Re(\lambda)>0$, $(|xz_1|^{\frac12}+\dots+|xz_k|^{\frac12})^2 + |z_{k+1}| + \dots + |z_n| < 1$.

(7.3.33) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}F_{AD}^{(n)}[a,b_1,\dots,b_n; \lambda, c_{k+1},\dots,c_n; z_1x,\dots,z_kx, z_{k+1},\dots,z_n]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \, {}^{(k)}F^{(n)}_{AD} \left[a, b_1, \ldots, b_n; \mu, c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right]$$

$Re(\lambda) > 0 \quad , \quad \max(|z_1 x|, \ldots, |z_k x|) + |z_{k+1}| + \ldots + |z_n| < 1 \quad .$

$$(7.3.34) \quad D_x^{\lambda-\mu} \left\{ x^{\lambda-1} \, {}^{(k)}F^{(n)}_{BD} \left[\mu, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n; c; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right] \right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \, {}^{(k)}F^{(n)}_{BD} \left[\lambda, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n; c; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right],$$

$Re(\lambda) > 0 \quad , \quad \max(|x z_1|, \ldots, |x z_k|, |z_{k+1}|, \ldots, |z_n|) < 1 \quad .$

$$(7.3.35) \quad D_x^{\lambda-\mu} \left\{ x^{\lambda-1} \, {}^{(k)}_{(1)}\Phi^{(n)}_{AC} \left[a, \mu; c_1, \ldots, c_n; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right] \right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \, {}^{(k)}_{(1)}\Phi^{(n)}_{AC} \left[a, \;; c_1, \ldots, c_n; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right], \; Re(\lambda) > 0$$

$$(7.3.36) \quad D_x^{\lambda-\mu} \left\{ x^{\lambda-1} \, {}^{(k)}_{(1)}\Phi^{(n)}_{AD} \left[a, b_1, \ldots, b_n; \lambda; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right] \right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \, {}^{(k)}_{(1)}\Phi^{(n)}_{AD} \left[a, b_1, \ldots, b_n; \mu; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right],$$

$Re(\lambda) > 0 \quad .$

$$(7.3.37) \quad D_x^{\lambda-\mu} \left\{ x^{\lambda-1} \, {}^{(k)}_{(1)}\Phi^{(n)}_{BD} \left[\mu, b_1, \ldots, b_n; c; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right] \right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \, {}^{(k)}_{(1)}\Phi^{(n)}_{BD} \left[\lambda, b_1, \ldots, b_n; c; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n \right], \; Re(\lambda) > 0$$

$$(7.3.38) \quad D_x^{\lambda-\mu} \left\{ x^{\lambda-1} \, {}^{(k)}F^{(n)}_{CD} \left[\mu, b, b_1, \ldots, b_k; c; c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_n x \right] \right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \, {}^{(k)}F^{(n)}_{CD} \left[\lambda, b, b_1, \ldots, b_k; c; c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_n x \right],$$

$Re(\lambda) > 0 \; , \quad \max(|z_1 x|, \ldots, |z_k x|) + (|z_{k+1}|^{\frac{1}{2}} + \ldots + |z_n|^{\frac{1}{2}})^2 < 1 \; ,$

(7.3.39) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\,{}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;\lambda,c_{k+1},\ldots,c_n;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}\,{}^{(k)}F^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;\mu;c_{k+1},\ldots,c_n;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right],$$

$\mathrm{Re}(\lambda)>0$, $\max(|z_1x|,\ldots,|z_kx|)+(|z_{k+1}|^{\frac{1}{2}}+\ldots+|z_n|^{\frac{1}{2}})^2<1$.

(7.3.40) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\,{}^{(k)}F^{(n)}_{CD}\left[a,\mu,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;z_1,\ldots,z_k,z_{k+1}x,\ldots,z_nx\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}\,{}^{(k)}F^{(n)}_{CD}\left[a,\lambda,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;z_1,\ldots,z_k,z_{k+1}x,\ldots,z_nx\right],$$

$\mathrm{Re}(\lambda)>0$, $\max(|z_1|,\ldots,|z_k|)+(|z_{k+1}x|^{\frac{1}{2}}+\ldots+|z_nx|^{\frac{1}{2}})^2<1$.

(7.3.41) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\,{}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[\mu,b;c,c_{k+1},\ldots,c_n;z_1x,\ldots,z_nx\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}\,{}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[\lambda,b;c,c_{k+1},\ldots,c_n;z_1x,\ldots,z_nx\right],\quad \mathrm{Re}(\lambda)>0 .$$

(7.3.42) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\,{}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,\mu;c,c_{k+1},\ldots,c_n;z_1,\ldots,z_k,z_{k+1}x,\ldots,z_nx\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}\,{}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,\lambda;c,c_{k+1},\ldots,c_n;z_1,\ldots,z_k,z_{k+1}x,\ldots,z_nx\right],$$

$\mathrm{Re}(\lambda)>0$.

(7.3.43) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\,{}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b;\lambda,c_{k+1},\ldots,c_n;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}\,{}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a,b;\mu,c_{k+1},\ldots,c_n;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right],\quad \mathrm{Re}(\lambda)>0.$$

(7.3.44) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\,{}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[\mu,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;z_1x,\ldots,z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{CD}[\lambda, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_n x], \quad Re(\lambda) > 0 .$$

(7.3.45) $$D_x^{\lambda-\mu}\{x^{\lambda-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{CD}[a, b_1, \ldots, b_k; \lambda, c_{k+1}, \ldots, c_n; z_k x, \ldots, z_k x, z_{k+1}, \ldots, z_n]\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{CD}[a, b_1, \ldots, b_k; \mu, c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n],$$

$Re(\lambda) > 0$.

(7.3.46) $$D_x^{\lambda-\mu}\{x^{\lambda-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{CD}[\mu, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; z_1, \ldots, z_k, z_{k+1} x, \ldots, z_n x]\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{CD}[\lambda, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; z_1, \ldots, z_k, z_{k+1} x, \ldots, z_n x],$$

$Re(\lambda) > 0$.

(7.3.47) $$D_x^{\lambda-\mu}\{x^{\lambda-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{CD}[b, b_1, \ldots, b_k; \lambda, c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n]\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{CD}[b, b_1, \ldots, b_k; \mu, c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n],$$

$Re(\lambda) > 0$.

(7.3.48) $$D_x^{\lambda-\mu}\{x^{\lambda-1} \; {}^{(k)}_{(4)}\Phi^{(n)}_{CD}[\mu, b, b_1, \ldots, b_k; c; z_1 x, \ldots, z_n x]\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(4)}\Phi^{(n)}_{CD}[\lambda, b, b_1, \ldots, b_k; c; z_1 x, \ldots, z_n x], \quad Re(\lambda) > 0 .$$

(7.3.49) $$D_x^{\lambda-\mu}\{x^{\lambda-1} \; {}^{(k)}_{(4)}\Phi^{(n)}_{CD}[a, \mu, b_1, \ldots, b_k; c; z_1, \ldots, z_k, z_{k+1} x, \ldots, z_n x]\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(4)}\Phi^{(n)}_{CD}[a, \lambda, b_1, \ldots, b_k; c; z_1, \ldots, z_k, z_{k+1}, \ldots, z_n], \quad Re(\lambda) > 0 .$$

(7.3.50) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;\lambda;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;\mu;\ z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right],\ Re(\lambda)>0\ .$$

(7.3.51) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[\mu,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;z_1,\ldots,z_k,\ z_{k+1}x,\ldots,z_nx\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[\lambda,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right],$$

$Re(\lambda)>0$.

(7.3.52) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[\mu,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;z_1x,\ldots,z_nx\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\left[\lambda,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;z_1x,\ldots,z_nx\right],\ \ Re(\lambda)>0\ .$$

(7.3.53) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[\mu,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;z_1x,\ldots,z_nx\right]\right\}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\ x^{\mu-1}\ {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\left[\lambda,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;z_1x,\ldots,z_nx\right],\ Re(\lambda)>0\ .$$

Here ${}^{(k)}F^{(n)}_{CD}$ is Karlsson's multiple hypergeometric function and

${}^{(k)}_{(1)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(3)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(4)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(5)}\Phi^{(n)}_{CD}$ and ${}^{(k)}_{(6)}\Phi^{(n)}_{CD}$

are the confluent forms of Karlsson ${}^{(k)}F^{(n)}_{CD}$ introduced in the chapter V .

(7.3.54) $$D_x^{\lambda-\mu}\left\{x^{\lambda-1}\ {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[\mu,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;z_1x,\ldots,z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{AD} [\lambda, b_1, \ldots, b_n; c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_n x], \quad Re(\lambda) > 0 .$$

$$(7.3.55) \quad D_x^{\lambda-\mu} \{ x^{\lambda-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{BD} [\mu, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; c; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n] \}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{BD} [\lambda, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; c; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n],$$

$Re(\lambda) > 0$.

$$(7.3.56) \quad D_x^{\lambda-\mu} \{ x^{\lambda-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{BD} [a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; \lambda; z_1 x, \ldots, z_n x] \}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{BD} [a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; \mu; z_1 x, \ldots, z_n x], \quad Re(\lambda) > 0 .$$

$$(7.3.57) \quad D_x^{\lambda-\mu} \{ x^{\lambda-1} \; {}^{(k)}_{(1)}\Phi^{(n)}_{D} [\mu, b_1, \ldots, b_n; c; z_1 x, \ldots, z_n x] \}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(1)}\Phi^{(n)}_{D} [\lambda, b_1, \ldots, b_n; c; z_1 x, \ldots, z_n x], \quad Re(\lambda) > 0 .$$

$$(7.3.58) \quad D_x^{\lambda-\mu} \{ x^{\lambda-1} \; {}^{(k)}_{(1)}\Phi^{(n)}_{D} [a, b_1, \ldots, b_n; \lambda; z_1 x, \ldots, z_n x] \}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(1)}\Phi^{(n)}_{D} [a, b_1, \ldots, b_n; \mu; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n], \quad Re(\lambda) > 0$$

$$(7.3.59) \quad D_x^{\lambda-\mu} \{ x^{\lambda-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{D} [\mu, b_1, \ldots, b_n; c; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n] \}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{D} [\lambda, b_1, \ldots, b_n; c; z_1 x, \ldots, z_k x, z_{k+1}, \ldots, z_n], \quad Re(\lambda) > 0$$

$$(7.3.60) \quad D_x^{\lambda-\mu} \{ x^{\lambda-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{D} [a, b_1, \ldots, b_n; \lambda; z_1 x, \ldots, z_n x] \}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{D} [a, b_1, \ldots, b_n; \mu; z_1 x, \ldots, z_n x], \quad Re(\lambda) > 0 .$$

$$(7.3.61) \quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1} \; {}^{(k)}_{(1)}\Phi_C^{(n)}\left[\mu,b;c_1,\ldots,c_n;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(1)}\Phi_C^{(n)}\left[\lambda,b;c_1,\ldots,c_n;z_1x,\ldots,z_kx,z_{k+1},\ldots,z_n\right], \; Re(\lambda)>0 \, .$$

$$(7.3.62) \quad D_x^{\lambda-\mu}\left\{ x^{\lambda-1} \; {}^{(k)}_{(1)}\Phi_C^{(n)}\left[a,\mu;c_1,\ldots,c_n;z_1x,\ldots,z_nx\right]\right\},$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} \; {}^{(k)}_{(1)}\Phi_C^{(n)}\left[a,\lambda,c_1,\ldots,c_n;z_1x,\ldots,z_nx\right] \;, \; Re(\lambda)>0 \, .$$

Here ${}^{(k)}_{(2)}\Phi_{AD}^{(n)}$ and ${}^{(k)}_{(3)}\Phi_{BD}^{(n)}$ are confluent forms of Lauricella intermediate functions ${}^{(k)}F_{AD}^{(n)}$ and ${}^{(k)}F_{BD}^{(n)}$ of Chandel and Gupta [2] while ${}^{(k)}_{(1)}\Phi_D^{(n)}$, ${}^{(k)}_{(2)}\Phi_D^{(n)}$ are new confluent forms of ExtOn's multiple hypergeometric functions ${}^{(k)}_{(1)}E_D^{(n)}$, ${}^{(k)}_{(2)}E_D^{(n)}$ [6,7] and ${}^{(k)}_{(1)}\Phi_C^{(n)}$ is confluent forms of Chandel's ${}^{(k)}_{(1)}E_C^{(n)}$. These are all new confluent multiple hypergeometric functions, introduced in chapter V .

7.4. SPECIAL CASES OF (7.2.2)

Use of two fractional derivative operators . Specializing the parameters in (7.2.2) , we obtain the following results :

$$(7.4.1) \quad D_x^{\lambda-\mu} D_y^{\lambda'-\mu'}\left\{ x^{\lambda-1} y^{\lambda'-1} \; {}^{(k)}F_{AC}^{(n)}\left[\mu,\mu',b_{k+1},\ldots,b_n;c_1,\ldots,c_n;z_1xy,\right.\right.$$

$$\left.\left. z_kxy, z_{k+1}x,\ldots, z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} \, {}^{(k)}F^{(n)}_{AC} [\lambda,\lambda',b_{k+1},\ldots,b_n;c_1,\ldots,c_n;z_1xy,\ldots,z_kxy,$$
$$z_{k+1}x,\ldots,z_nx] ,$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $(|z_1xy|^{\frac{1}{2}} +\ldots+ |z_kxy|^{\frac{1}{2}})^2 + |z_{k+1}x|+\ldots+|z_nx| < 1$.

(7.4.2) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \{ x^{\lambda-1} y^{\lambda'-1} \, {}^{(k)}F^{(n)}_{AD} [\mu,b_1,\ldots,b_n;\lambda',c_{k+1},\ldots,c_n;z_1xy,$$
$$\ldots,z_kxy, z_{k+1}x,\ldots,z_nx] \}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} \, {}^{(k)}F^{(n)}_{AD} [\lambda,b_1,\ldots,b_n:\mu',c_{k+1},\ldots,c_n:z_1xy,\ldots,z_kxy,$$
$$z_{k+1}x ,\ldots,z_nx] ,$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $\max(|z_1xy| ,\ldots,|z_kxy|,|z_{k+1}x| ,\ldots, |z_nx|) < 1$.

(7.4.3) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \{ x^{\lambda-1} y^{\lambda'-1} \, {}^{(k)}F^{(n)}_{BD} [\mu',a_{k+1},\ldots,a_n,b_1,\ldots,b_n;\lambda;$$
$$z_1xy,\ldots,z_kxy,z_{k+1}x, \ldots, z_nx] \}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} \, {}^{(k)}F^{(n)}_{BD} [\lambda',a_{k+1},\ldots,a_n,b_1,\ldots,b_n;\mu;z_1xy,\ldots,z_kxy,$$
$$z_{k+1}x,\ldots,z_nx] ,$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $\max(|z_1xy| ,\ldots, |z_kxy|,|z_{k+1}x| \ldots |z_nx|) < 1$.

(7.4.4) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \{ x^{\lambda-1} y^{\lambda'-1} \, {}^{(k)}F^{(n)}_{CD} [\mu,b,b_1,\ldots,b_k;\lambda',c_{k+1},\ldots,c_n;z_1xy,$$
$$\ldots,z_kxy,z_{k+1}x,\ldots,z_nx] \}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} \, {}^{(k)}F^{(n)}_{CD} [\lambda,b,b_1,\ldots,b_k;\mu',c_{k+1},\ldots,c_n;z_1xy,\ldots,z_kxy,$$
$$z_{k+1}x,\ldots,z_nx] ,$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $\max(|z_1xy| ,\ldots, |z_kxy|) + (|z_{k+1}y|^{\frac{1}{2}} +\ldots+ |z_nx|^{\frac{1}{2}}) < 1$.

(7.4.5) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \{ x^{\lambda-1} y^{\lambda'-1} \, {}^{(k)}F^{(n)}_{CD} [\mu,\mu',b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;$$
$$z_1x,\ldots,z_kx,z_{k+1}y,\ldots,z_ny] \}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\; {}^{(k)}F^{(n)}_{CD}\left[\lambda, \lambda', b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_k x, z_{k+1} xy, \ldots, z_n xy\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $\max(|z_1 x|, \ldots, |z_k x|)+(|z_{k+1} xy|^{\frac{1}{2}} + \ldots + |z_n x|^{\frac{1}{2}})^2 < 1$.

$$(7.4.6)\quad D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1}\, y^{\lambda'-1}\; {}^{(k)}F^{(n)}_{CD}\left[a, \mu'; b_1, \ldots, b_k; \lambda, c_{k+1}, \ldots, c_n; z_1 x, z_k x, z_{k+1} y, \ldots, z_n y\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\; {}^{(k)}F^{(n)}_{CD}\left[a, \lambda', b_1, \ldots, b_k; \mu, c_{k+1}, \ldots, c_n; z_1 x, \ldots, z_k x, z_{k+1} y, \ldots, z_n y\right],$$

$Re(\lambda) > 0$, $Re(\lambda')>0$, $\max(|z_1 x|, \ldots, |z_k x|)+(|z_{k+1} y|^{\frac{1}{2}} + \ldots |z_n y|^{\frac{1}{2}})^2 < 1$.

$$(7.4.7)\quad D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1}\, y^{\lambda'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{AC}\left[\mu, \mu'; c_1, \ldots, c_n; z_1 xy, \ldots, z_k xy, z_{k+1} x, \ldots, z_n x\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{AC}\left[\lambda, \lambda'; c_1, \ldots, c_n; z_1 xy, \ldots, z_k xy, z_{k+1} x, \ldots, z_n x\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

$$(7.4.8)\quad D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1}\, y^{\lambda'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{AD}\left[\mu, b_1, \ldots, b_n; \lambda'; z_1 xy, \ldots, z_k xy, z_{k+1} x, \ldots, z_n x\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{AD}\left[\lambda, b_1, \ldots, b_n; \mu'; z_1 xy, \ldots, z_k xy, z_{k+1} x, \ldots, z_n x\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

$$(7.4.9)\quad D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1}\, y^{\lambda'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{BD}\left[\mu', b_1, \ldots, b_n; \lambda; z_1 xy, \ldots, z_k xy, z_{k+1} x, \ldots, z_n x\right]\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} {}^{(k)}_{(1)}F^{(n)}_{BD} [\lambda', b_1, \ldots, b_n; \mu; z_1 xy, \ldots, z_k xy, z_{k+1} x, \ldots, z_n x],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

$$(7.4.10) \quad D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \{ x^{\lambda-1} y^{\lambda'-1} {}^{(k)}_{(1)}E^{(n)}_{D} [a, b_1, \ldots, b_n; \lambda, \lambda'; z_1 x, \ldots, z_k x, z_{k+1} y, \ldots, z_n y] \}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} {}^{(k)}_{(1)}E^{(n)}_{D} [a, b_1, \ldots, b_n; \mu, \mu'; z_1 x, \ldots, z_k x, z_{k+1} y, \ldots, z_n y],$$

$Re(\lambda)>0$, $Re(\lambda')>0$. $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_n$, $r_k + r_n = 1$, $|z_i x| < r_i$, $i=1,\ldots,k$. and $|z_i y| < r_i$, $i=k+1,\ldots,n$.

$$(7.4.11) \quad D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \{ x^{\lambda-1} y^{\lambda'-1} {}^{(k)}_{(1)}E^{(n)}_{D} [\mu, b_1, \ldots, b_n; \lambda, c; z_1 xy, \ldots, z_k xy, z_{k+1} x, \ldots, z_n x] \}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} {}^{(k)}_{(1)}E^{(n)}_{D} [\lambda, b_1, \ldots, b_n; \mu, c; z_1 xy, \ldots, z_k xy, z_{k+1} y, \ldots, z_n y],$$

$Re(\lambda)>0$, $Re(\lambda')>0$. $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_n$, $r_k + r_n = 1$, $|z_i xy| < r_i$,

$i=1,\ldots,k$. and $|z_i y| < r_i$, $i=k+1,\ldots,n$.

$$(7.4.12) \quad D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \{ x^{\lambda-1} y^{\lambda'-1} {}^{(k)}_{(1)}E^{(n)}_{D} [\mu, b_1, \ldots, b_n; c, \lambda'; z_1 x, \ldots, z_k x, z_{k+1} yx, \ldots, z_n yx] \}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} {}^{(k)}_{(1)}E^{(n)}_{D} [\lambda, b_1, \ldots, b_n; c, \mu'; z_1 x, \ldots, z_k x, z_{k+1} xy, \ldots, z_n xy],$$

$Re(\lambda) > 0$, $Re(\lambda') > 0$, $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_n$, $r_k + r_n = 1$, $|z_i x| < r_i$,

$i=1, \ldots, k$, and $|z_i xy| < r_i$, $i=k+1, \ldots, n$.

$$(7.4.13) \quad D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \{ x^{\lambda-1} y^{\lambda'-1} {}^{(k)}_{(2)}E^{(n)}_{D} [\mu, \mu', b_1, \ldots, b_n; c; z_1 x, \ldots, z_k x, z_{k+1} y, \ldots, z_n y] \}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\; x^{\mu-1}\; y^{\mu'-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}\left[\lambda,\lambda',b_1,\ldots,b_n;c;z_1x,\ldots,z_kx,z_{k+1}y,\ldots,z_ny\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_k\cdot r_n=r_k+r_n$;

$|z_ix|<r_i$, $i=1,\ldots,k$ $|z_iy|<r_i$, $i=k+1,\ldots,n$.

(7. .14) $$D_x^{\lambda'-\mu}\; D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}\; y^{\lambda'-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}\left[a,\mu',b_1,\ldots,b_n;\lambda;z_1x,\ldots,z_kx,\; z_{k+1}xy,\ldots,z_nxy\right]\right\}$$

$$= \frac{\Gamma(\lambda)\;\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\; x^{\mu-1}\; y^{\mu'-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}\left[a,\lambda',b_1,\ldots,b_n;\mu;z_1x,\ldots,z_kx,z_{k+1}xy,\ldots,z_ny\right]$$

$Re(\lambda)>0$, $Re(\lambda')>0$. $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_k\cdot r_n=r_k+r_n$

$|z_ix_i|<r_i$, $i=1,\ldots,k$; $|z_ixy|<r_i$, $i=k+1,\ldots,n$.

(7.4.15) $$D_x^{\lambda-\mu}\; D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}\; y^{\lambda'-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}\left[\mu',a',b_1,\ldots,b_n;\lambda;z_1xy,\ldots,z_kxy,\; z_{k+1}x,\ldots,z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\; x^{\mu-1}\; y^{\mu'-1}\; {}^{(k)}_{(2)}E^{(n)}_{D}\left[\lambda',a',b_1,\ldots,b_n;\mu;z_1xy,\ldots,z_kxy,z_{k+1}x,\ldots,z_nx\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_k\cdot r_n=r_k+r_n$,

$|z_ixy|<r_i$, $i=1,\ldots,k$ and $|z_ix|<r_i$, $j=k+1,\ldots,n$.

(7.4.16) $$D_x^{\lambda-\mu}\; D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}\; y^{\lambda'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[\mu,\mu';c,c_{k+1},\ldots,c_n;z_1x,\ldots,z_kx,\; z_{k+1}xy,\ldots,z_nxy\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\; x^{\mu-1}\; y^{\mu'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[\lambda,\lambda';c_{k+1},\ldots,c_n;z_1x,\ldots,z_kx,z_{k+1}xy,\ldots,z_nxy\right]$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

$$(7.4.17)\quad D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\left\{ x^{\lambda-1}\, y^{\lambda'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[\mu, b; \lambda', c_{k+1}, \ldots, c_n; z_1xy, \ldots, z_kxy, z_{k+1}x, \ldots, z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[\lambda, b; \mu', c_{k+1}, \ldots, c_n; z_1xy, \ldots, z_kxy, z_{k+1}x, \ldots, z_nx\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

$$(7.4.18)\quad D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\left\{ x^{\lambda-1}\, y^{\lambda'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a, \mu; \lambda', c_{k+1}, \ldots, c_n; z_1y, \ldots, z_ky, z_{k+1}x, \ldots, z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\; {}^{(k)}_{(1)}\Phi^{(n)}_{CD}\left[a, \lambda; \mu', c_{k+1}, \ldots, c_n; z_1y, \ldots, z_ky, z_{k+1}x, \ldots, z_nx\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

$$(7.4.19)\quad D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\left\{ x^{\lambda-1}\, y^{\lambda'-1}\; {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[\mu, b_1, \ldots, b_k; \lambda', c_{k+1}, \ldots, c_n; z_1xy, \ldots, z_kxy, z_{k+1}x, \ldots, z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\; {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[\lambda, b_1, \ldots, b_k; \mu', c_{k+1}, \ldots, c_n; z_1xy, \ldots, z_kxy, z_{k+1}x, \ldots, z_nx\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

$$(7.4.20)\quad D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\left\{ x^{\lambda-1}\, y^{\lambda'-1}\; {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[\mu, b_1, \ldots, b_k; \lambda', c_{k+1}, \ldots, c_n; z_1y, \ldots, z_ky, z_{k+1}x, \ldots, z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\; {}^{(k)}_{(3)}\Phi^{(n)}_{CD}\left[\lambda, b_1, \ldots, b_k; \mu', c_{k+1}, \ldots, c_n; z_1y, \ldots, z_ky, z_{k+1}x, \ldots, z_nx\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

(7.4.21) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \{ x^{\lambda-1}\, y^{\lambda'-1}\, {}^{(k)}_{(1)}\Phi^{(n)}_{CD} [\mu,\mu',b_1,\ldots,b_k;c;xz_1,\ldots,xz_k, xyz_{k+1},\ldots,z_n xy] \}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')} x^{\mu-1}\, y^{\mu'-1}\, {}^{(k)}_{(4)}\Phi^{(n)}_{CD} [\lambda,\lambda',b_1,\ldots,b_k;c;xz_1,\ldots,xz_k,xyz_{k+1},\ldots,xyz_n],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

(7.4.22) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \{ x^{\lambda-1}\, y^{\lambda'-1}\, {}^{(k)}_{(4)}\Phi^{(n)}_{CD} [\mu,b,b_1,\ldots,b_k;\lambda';z_1 xy,\ldots,z_k xy, z_{k+1}x,\ldots,z_n x] \}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')} x^{\mu-1}\, y^{\mu'-1}\, {}^{(k)}_{(4)}\Phi^{(n)}_{CD} [\lambda,b,b_1,\ldots,b_k;\mu';z_1 xy,\ldots,z_k xy,z_{k+1}x,\ldots,z_n x],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

(7.4.23) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \{ x^{\lambda-1}\, y^{\lambda'-1}\, {}^{(k)}_{(4)}\Phi^{(n)}_{CD} [a,\mu,b_1,\ldots,b_k;\lambda';z_1 y,\ldots,z_k y, z_{k+1}x,\ldots,z_n x] \}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')} x^{\mu-1}\, y^{\mu'-1}\, {}^{(k)}_{(4)}\Phi^{(n)}_{CD} [a,\lambda,b_1,\ldots,b_k;\mu';z_1 y,\ldots,z_k y,z_{k+1}x,\ldots,z_n x],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

(7.4.24) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \{ x^{\lambda-1}\, y^{\lambda'-1}\, {}^{(k)}_{(5)}\Phi^{(n)}_{CD} [\mu,\mu',b_1,\ldots,b_k;c_{k+1},\ldots,c_n;z_1 x,\ldots, z_k x,z_{k+1}xy,\ldots,z_n xy] \}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')} x^{\mu-1}\, y^{\mu'-1}\, {}^{(k)}_{(5)}\Phi^{(n)}_{CD} [\lambda,\lambda',b_1,\ldots,b_k;c_{k+1},\ldots,c_n;z_1 x,\ldots,z_k x, z_{k+1}xy,\ldots,z_n xy],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

(7.4.25) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'} \{ x^{\lambda-1}\, y^{\lambda'-1}\, {}^{(k)}_{(3)}\Phi^{(n)}_{BD} [\mu,a_{k+1},\ldots,a_n,b_1,\ldots,b_k;\lambda';z_1 xy, \ldots,z_k xy,z_{k+1}y,\ldots,z_n y] \}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\, {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[\lambda, a_{k+1},\ldots,a_n, b_1,\ldots,b_k;\mu'; z_1xy,\ldots,z_kxy,\; z_{k+1}y,\ldots z_ny\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

(7.4.26) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}\, y^{\lambda'-1}\, {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[\mu, b_1,\ldots,b_n;\lambda'; z_1xy,\ldots,z_kxy,\; z_{k+1}x,\ldots,z_nx\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\, {}^{(k)}_{(1)}\Phi^{(n)}_{D}\left[\lambda, b_1,\ldots,b_n;\mu'; z_1xy,\ldots,z_kxy, z_{k+1},\ldots,z_nx\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

(7.4.27) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}\, y^{\lambda'-1}\, {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[\mu, b_1,\ldots,b_n;\lambda'; z_1xy,\ldots,z_kxy,\; z_{k+1}y,\ldots,z_ny\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\, {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[\lambda, b_1,\ldots,b_n;\mu'; z_1xy,\ldots,z_kxy, z_{k+1}y,\ldots,z_ny\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$.

(7.4.28) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}\, y^{\lambda'-1}\, {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[\mu,\mu', c_1,\ldots,c_n; z_1xy,\ldots,z_kxy,\; z_{k+1}y,\ldots,z_ny\right]\right\}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')}{\Gamma(\mu)\,\Gamma(\mu')}\, x^{\mu-1}\, y^{\mu'-1}\, {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[\lambda,\lambda', c_1,\ldots,c_n; z_1xy,\ldots,z_kxy, z_{k+1}y,\ldots,z_ny\right]$$

$Re(\lambda)>0$, $Re(\lambda)>0$.

7.5. Use of three fractional derivative operators

In this section, we obtain the following operational relationships :

(7.5.1) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\, D_z^{\lambda''-\mu''} \Big\{ x^{\lambda-1}\, y^{\lambda'-1}\, z^{\lambda''-1} \cdot {}^{(k)}_{(1)}E^{(n)}_{C} \left[\mu,\mu',\mu'';b_1,\ldots,b_n;\xi_1 xy,\ldots,\xi_k xy,\xi_{k+1} xz,\ldots,\xi_n xz\right] \Big\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}\, x^{\mu-1}\, y^{\mu'-1}\, z^{\mu''-1}\; {}^{(k)}_{(1)}E^{(n)}_{C} \left[\lambda,\lambda',\lambda'';b_1,\ldots,b_n;\xi_1 xy,\ldots,\xi_k xy,\xi_{k+1} xz,\ldots,\xi_n xz\right],$$

$\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\lambda')>0$, $\mathrm{Re}(\lambda'')>0$, $(\sqrt{r_1}+\ldots+\sqrt{r_k})^2+(\sqrt{r_{k+1}}+\ldots+\sqrt{r_n})^2=1$

$|\xi_i xy|<r_i$, $i=1,\ldots,k$; $|\xi_i xz|<r_i$, $i=k+1,\ldots,n$.

(7.5.2) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\, D_z^{\lambda''-\mu''} \Big\{ x^{\lambda'-1}\, y^{\lambda'-1}\, z^{\lambda''-1} \cdot {}^{(k)}_{(1)}E^{(n)}_{D} \left[\mu,b_1,\ldots,b_n;\lambda',\lambda'';\xi_1 xy,\ldots,\xi_k xy,\xi_{k+1} xz,\ldots,\xi_n xz\right] \Big\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}\, x^{\mu-1}\, y^{\mu'-1}\, z^{\mu''-1}\; {}^{(k)}_{(1)}E^{(n)}_{D} \left[\lambda,b_1,\ldots,b_n;\mu,\mu';\xi_1 xy,\ldots,\xi_k xy,\xi_{k+1} xz,\ldots,\xi_n xz\right],$$

$\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\lambda')>0$, $\mathrm{Re}(\lambda'')>0$. $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_k+r_n=1$,

$|\xi_i xy|<r_i$, $i=1,\ldots,k$ and $|\xi_i xz|<r_i$ $i=k+1,\ldots,n$.

(7.5.3) $$D_x^{\lambda-\mu}\, D_y^{\lambda'-\mu'}\, D_z^{\lambda''-\mu''} \Big\{ x^{\lambda-1}\, y^{\lambda'-1}\, z^{\lambda''-1} \cdot {}^{(k)}_{(2)}E^{(n)}_{D} \left[\mu',\mu'',b_1,\ldots,b_n;\lambda;\xi_1 xy,\ldots,\xi_k xy,\xi_{k+1} xz,\ldots,\xi_n xz\right] \Big\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}\, x^{\mu-1}\, y^{\mu'-1}\, z^{\mu'-1}\; {}^{(k)}_{(2)}E^{(n)}_{D} \left[\lambda',\lambda'',b_1,\ldots,b_n;\mu;\xi_1 xy,\ldots,\xi_k xy,\xi_{k+1} xz,\ldots,\xi_n xz\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $Re(\lambda'')>0$, $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_k+r_n=r_n \cdot r_k$,

$|\xi_i xy| < r_i$, $i=1,\ldots,k$. and $|\xi_i xz| < r_i$, $i=k+1,\ldots,n$.

(7.5.4) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \{ x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} \cdot$$

$$^{(k)}F_{CD}^{(n)} [\mu,\mu',b_1,\ldots,b_k;\lambda'',c_{k+1},\ldots,c_n;\xi_1 xz,\ldots,\xi_k xz,\xi_{k+1} xy,\ldots,\xi_n xy] \}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} \, {}^{(k)}F_{CD}^{(n)} [\lambda,\lambda',b_1,\ldots,b_k;\mu'',c_{k+1},\ldots,c_n;\xi_1 xz,\ldots, \xi_k xz,\xi_{k+1} xy,\ldots,\xi_n xy] ,$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $Re(\lambda'')>0$,

$\max(|\xi_1 xz| ,\ldots, |\xi_k xz|) + (|\xi_{k+1} xy|^{\frac{1}{2}} + \ldots + |\xi_n xy|^{\frac{1}{2}})^2 < 1$.

(7.5.5) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \{ x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} \cdot$$

$$^{(k)}_{(1)}\Phi_{CD}^{(n)} [\mu,\mu';\lambda'',c_{k+1},\ldots,c_n;\xi_1 xz,\ldots,\xi_k xz,\xi_{k+1} xy,\ldots,\xi_n xy] \}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} \, {}^{(k)}_{(1)}\Phi_{CD}^{(n)} [\lambda,\lambda';\mu'',c_{k+1},\ldots,c_n;\xi_1 xz,\ldots,\xi_k zx, \xi_{k+1} xy,\ldots,\xi_n xy] ,$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $Re(\lambda'')>0$.

(7.5.6) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \{ x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} \cdot$$

$$^{(k)}_{(4)}\Phi_{CD}^{(n)} [\mu,\mu',b_1,\ldots,b_k;\lambda'';\xi_1 xz,\ldots,\xi_k xz,\xi_{k+1} xy,\ldots,\xi_n xy] \}$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\lambda')\,\Gamma(\lambda'')}{\Gamma(\nu)\,\Gamma(\mu)\,\Gamma(\mu')}\; x^{\mu-1}\; y^{\mu'-1}\; z^{\mu''-1}\; {}^{(k)}_{(4)}\Phi^{(n)}_{CD}\left[\lambda, \lambda', b_1, \ldots, b_k; \mu''; \xi_1 xz, \ldots, \xi_k xz, \xi_{k+1} xy, \ldots, \xi_n xy\right],$$

$Re(\lambda)>0$, $Re(\lambda')>0$, $Re(\lambda'')>0$.

7.6 MULTIDIMENSIONAL FRACTIONAL DERIVATIVES

In this section, we derive the following multidimensional fractional derivatives involving the above multiple hypergeometric functions :

$$(7.6.1)\quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j-1}\; F_A^{(n)}\left[a, \mu_1, \ldots, \mu_n; c_1, \ldots, c_n; z_1 x_1, \ldots, z_n x_n\right]\right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\; x_j^{\mu_j-1}\; F_A^{(n)}\left[a, \lambda_1, \ldots, \lambda_n; c_1, \ldots, c_n; z_1 x_1, \ldots, z_n x_n\right],$$

$Re(\lambda_i) > 0$, $i=1,\ldots, n$. $|z_1 x_1| + \ldots + |z_n x_n| < 1$.

$$(7.6.2)\quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j-1} .\; F_A^{(n)}\left[a, b_1, \ldots, b_n; \lambda_1, \ldots, \lambda_n; z_1 x_1, \ldots, z_n x_n\right]\right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\; x_j^{\mu_j-1}\; F_A^{(n)}\left[a, b_1, \ldots, b_n; \mu_1, \ldots, \mu_n; z_1 x_1, \ldots, z_n x_n\right],$$

$|z_1 x_1| + \ldots + |z_n x_n| < 1$, $Re(\lambda_j) > 0$, $i=1, \ldots , n$.

$$(7.6.3)\quad D_{x_1}^{\lambda_1-\mu_1}\ \dots\ D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\,.\right.$$

$$\left.F_B^{(n)}[\mu_1,\dots,\mu_n,b_1,\dots,b_n;c;z_1x_1,\ \dots,\ z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\quad F_B^{(n)}[\lambda_1,\dots,\lambda_n;b_1,\dots,b_n;c;z_1x_1,\dots,z_nx_n],$$

$$\max\{|z_1x_1|,\dots,|z_nx_n|\}<1\ ,\quad \mathrm{Re}(\lambda_j)>0\ ,\quad j=1,\dots,n\ .$$

$$(7.6.4)\quad D_{x_1}^{\lambda_1-\mu_1}\ \dots\ D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\ F_B^{(n)}[a_1,\dots,a_n,\mu_1,\dots,\mu_n;c;z_1x_1,\dots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\quad F_B^{(n)}[a_1,\dots,a_n,\lambda_1,\dots,\lambda_n;c;z_1x_1,\dots,z_nx_n]\ ,$$

$$\max\{|z_1x_1|^{\frac{1}{2}},\dots,|z_nx_n|^{\frac{1}{2}}\}<1\ ,\quad \mathrm{Re}(\lambda_j)>0\ ,\quad j=1,\dots,n\ .$$

$$(7.6.5)\quad D_{x}^{\lambda_1-\mu_1}\dots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n}x_j^{\lambda_j-1}F_C^{(n)}[a,b;\lambda_1,\dots,\lambda_n;z_1x_1,\dots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\quad F_C^{(n)}[a,b;\mu_1,\dots,\mu_n;z_1x_1,\ \dots\ ,z_nx_n]\ ,$$

$$|z_1x_1|^{\frac{1}{2}}+\dots+|z_nx_n|^{\frac{1}{2}}<1\ ,\quad \mathrm{Re}(\lambda_i)>0\ ,\quad i=1,\dots,n\ .$$

$$(7.6.6)\quad D_{x_1}^{\lambda_1-\mu_1}\ \dots\ D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n}x_j^{\lambda_j-1}F_D^{(n)}[a,\mu_1,\dots,\mu_n;c;z_1x_1,\dots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\quad F_D^{(n)}[a,\lambda_1,\dots,\lambda_n;c;\ z_1x_1,\dots,z_nx_n]\ ,$$

$$\max\{|z_1x_1|,\dots,|z_nx_n|\}<1\ ,\quad \mathrm{Re}(\lambda_i)>0\ ,\quad i=1,\dots,n\ .$$

$$(7.6.7)\quad D_{x_1}^{\lambda_1-\mu_1}\dots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\,{}_{(1)}^{(k)}E_D^{(n)}[a,\mu_1,\dots,\mu_n;c,c;z_1x_1,\dots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\,{}_{(1)}^{(k)}E_D^{(n)}[a,\lambda_1,\dots,\lambda_n;c,c';\ z_1x_1,\dots,z_nx_n]\ ,$$

$r_1=\dots=r_k,\ r_{k+1}=\dots=r_n\ ,\quad r_k+r_n=1\ ,\quad \mathrm{Re}(\lambda_i)>0\ ,\quad i=1,\dots,n\ .$

$$(7.6.8)\quad D_{x_1}^{\lambda_1-\mu_1}\dots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\,{}_{(2)}^{(k)}E_D^{(n)}[a,a',\mu_1,\dots,\mu_n;c;z_1x_1,\dots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\,{}_{(2)}^{(k)}E_D^{(n)}[a,a',\lambda_1,\dots,\lambda_n;c;\ z_1x_1,\dots,z_nx_n]\ ,$$

$r_1=\dots=r_k\ ,\quad r_{k+1}=\dots=r_n\ ,\ r_k\cdot r_n=r_k+r_n\ ,\quad \mathrm{Re}(\lambda_i)>0\ ,\quad i=1,\dots,n.$

$|z_ix_i|<r_i$

$$(7.6.9)\quad D_{x_1}^{\lambda_1-\mu_1}\dots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\,{}_{(1)}^{(k)}E_C^{(n)}[a,a',b;\lambda_1,\dots,\lambda_n;z_1x_1,\dots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\cdot x_j^{\mu_j-1}\,{}_{(1)}^{(k)}E_C^{(n)}[a,a',b;\mu_1,\dots,\mu_n;z_1x_1,\dots,z_nx_n]\ ,$$

$|z_ix_i|<r_i\ ,\ i=1,\dots,n;\quad (\sqrt{r_1}+\dots+\sqrt{r_k})^2+(\sqrt{r_{k+1}}+\dots+\sqrt{r_n})^2=1$

$\mathrm{Re}(\lambda_i)>0\ .$

$$(7.6.10)\quad D_{x_1}^{\lambda_1-\mu_1}\dots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\,{}^{(k)}F_{AC}^{(n)}[a,b,b_{k+1},\dots,b_n;\lambda_1,\dots,\lambda_n;z_1x_1,\dots,z_nx_n]\right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} \; x_j^{\mu_j - 1} \quad {}^{(k)}F^{(n)}_{AC} [a, b, b_{k+1}, \ldots, b_n; \mu_1, \ldots, \mu_n; z_1 x_1, \ldots, z_n x_n] ,$$

$(|z_1 x|^{\frac{1}{2}} + \ldots + |z_k x|^{\frac{1}{2}})^2 + |z_{k+1} x| + \ldots + |z_n x| < 1, \quad Re(\lambda_i) > 0 , \quad i=1,\ldots,n.$

$$(7.6.11) \quad D_{x_{k+1}}^{\lambda_{k+1} - \mu_{k+1}} \ldots D_{x_n}^{\lambda_n - \mu_n} \Big\{ \prod_{j=k+1}^{n} x_j^{\lambda_j - 1} \cdot$$

$${}^{(k)}F^{(n)}_{AC} [a, b, \mu_{k+1}, \ldots, \mu_n, c_1, \ldots, c_n; z_1, \ldots, z_k, z_{k+1} x_{k+1}, \ldots, z_n x_n] \Big\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} \cdot x_j^{\mu_j - 1} \; {}^{(k)}F^{(n)}_{AC} [a, b, \lambda_{k+1}, \ldots, \lambda_n, c_1, \ldots, c_n; z_1, \ldots, z_k, z_{k+1} x_{k+1}, \ldots, z_n x_n] ,$$

$(|z_1 x|^{\frac{1}{2}} + \ldots + |z_k x|^{\frac{1}{2}})^2 + |z_{k+1} x| + \ldots + |z_n x| < 1, \quad Re(\lambda_i) > 0 , \quad i=k+1,\ldots,n .$

$$(7.6.12) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \Big\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} \cdot$$

$${}^{(k)}F^{(n)}_{AD} [a, \mu_1, \ldots, \mu_n; c', c_{k+1}, \ldots, c_n; z_1 x_1, \ldots, z_n x_n] \Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} \cdot x_j^{\mu_j - 1} \; {}^{(k)}F^{(n)}_{AD} [a, \lambda_1, \ldots, \lambda_n; c, c_{k+1}, \ldots, c_n; z_1 x_1, \ldots, z_n x_n],$$

$\max \; |z_1 x_1| \; , \ldots , \; |z_k x_k| \; + |z_{k+1} x_{k+1}| + \ldots + |z_n x_n| < 1, \quad Re(\lambda_i) > 0 ,$

$i=1,\ldots,n .$

$$(7.6.13) \quad D_{x_{k+1}}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \Big\{ \prod_{j=k+1}^{n} x_j^{\lambda_j - 1} \cdot$$

$${}^{(k)}F^{(n)}_{AD} [a, b_1, \ldots, b_n; c; \lambda_{k+1}, \ldots, \lambda_n; z_1, \ldots, z_k, z_{k+1} x_{k+1}, \ldots, z_n x_n] \Big\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x^{\mu_j - 1} \, {}^{(k)}F_{AD}^{(n)}[a, b_1, \ldots, b_n; c, \mu_{k+1}, \ldots, \mu_n; z_1, \ldots, z_k, z_{k+1}x_{k+1}, \ldots, z_n x_n],$$

$\max\{|z_1|, \ldots, |z_k|\} + |z_{k+1}x_{k+1}| + \ldots + |z_n x_n| < 1$, $Re(\lambda_i) > 0$, $i = 1, \ldots, n$.

$$(7.6.14) \quad D_{x_{k+1}}^{\lambda_{k+1} - \mu_{k+1}} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=k+1}^{n} x_j^{\lambda_j - 1} \cdot {}^{(k)}F_{BD}^{(n)}[a, \mu_{k+1}, \ldots, \mu_n, b_1, \ldots, b_n; c; z_1, \ldots, z_k, z_{k+1}x_{k+1}, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, {}^{(k)}F_{BD}^{(n)}[a, \lambda_{k+1}, \ldots, \lambda_n, b_1, \ldots, b_n; c; z_1, \ldots, z_k, z_{k+1}x_{k+1}, \ldots, z_n x_n],$$

$\max\{|z_1|, \ldots, |z_k|, |z_{k+1}x_{k+1}|, \ldots, |z_n x_n|\} < 1$, $Re(\lambda_i) > 0$, $i = k+1, \ldots, n$.

$$(7.6.15) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} \cdot {}^{(k)}F_{BD}^{(n)}[a, a_{k+1}, \ldots, a_n, \mu_1, \ldots, \mu_n; c; z_1 x_1, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} \cdot x_j^{\mu_j - 1} \, {}^{(k)}F_{BD}^{(n)}[a, a_{k+1}, \ldots, a_n, \lambda_1, \ldots, \lambda_n; c; z_1 x_1, \ldots, z_n x_n],$$

$\max\{|z_1 x_1|, \ldots, |z_n x_n|, |z_{k+1}x_{k+1}|, \ldots, |z_n x_n|\} < 1$, $Re(\lambda_i) > 0$, $i = 1, \ldots, n$.

$$(7.6.16) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} \cdot \right.$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x^{\mu_j - 1} \, {}^{(k)}F_{CD}^{(n)} [a, b, \lambda_1, \ldots, \lambda_k; c; \mu_{k+1}, \ldots, \mu_n; z_1 x_1, \ldots, z_n x_n],$$

$\mathrm{Re}(\lambda_i) > 0$, $i = 1, \ldots, n$,

$\max(|z_1 x| , \ldots, |z_k x|) + (|z_{k+1}|^{\frac{1}{2}} + \ldots + |z_n|^{\frac{1}{2}})^2 < 1$.

$$(7.6.17) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} \cdot \Xi_1^{(n)} [\mu_1, \ldots, \mu_n, b_1, \ldots, b_{n-1}; c; z_1 x_1, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, \Xi_1^{(n)} [\lambda_1, \ldots, \lambda_n, b_1, \ldots, b_{n-1}; c; z_1 x_1, \ldots, z_n x_n],$$

$\mathrm{Re}(\lambda_i) > 0$, $i = 1, \ldots, n$.

$$(7.6.18) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_{n-1}}^{\lambda_{n-1} - \mu_{n-1}} \left\{ \prod_{j=1}^{n-1} x_j^{\lambda_j - 1} \Xi_1^{(n)} [a_1, \ldots, a_n, \mu_1, \ldots, \mu_{n-1}; c; z_1 x_1, \ldots, z_{n-1} x_{n-1}, z_n] \right\}$$

$$= \prod_{j=1}^{n-1} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, \Xi_1^{(n)} [a_1, \ldots, a_n, \lambda_1, \ldots, \lambda_{n-1}; c; z_1 x_1, \ldots, z_{n-1} x_{n-1}, z_n],$$

$\mathrm{Re}(\lambda_i) > 0$, $i = 1, \ldots, n-1$.

$$(7.6.19) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_{n-1}}^{\lambda_{n-1} - \mu_{n-1}} \left\{ \prod_{j=1}^{n-1} x_j^{\mu_j - 1} \cdot \Phi_3^{(n)} [\mu_1, \ldots, \mu_{n-1}; c; z_1 x_1, \ldots, z_{n-1} x_{n-1}, z_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, \Phi_3^{(n)} [\lambda_1, \ldots, \lambda_{n-1}; c; z_1 x_1, \ldots, z_{n-1} x_{n-1}, z_n],$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i=1 , \ldots , n-1 \quad .$

$$(7.6.20) \quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j-1} \Psi_2^{(n)} [a; \lambda_1, \ldots, \lambda_n; z_1 x_1, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \Psi_2^{(n)} [a; \mu_1, \ldots, \mu_n; z_1 x_1 , \ldots , z_n x_n] .$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i=1, \ldots , n \quad .$

$$(7.6.21) \quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j-1} \Phi_2^{(n)} [\mu_1, \ldots, \mu_n; c; z_1 x_1, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \cdot \Phi_2^{(n)} [\lambda_1, \ldots, \lambda_n; c; z_1 x_1, \ldots, z_n x_n] ,$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i = 1, \ldots , n$

$$(7.6\ 22) \quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_{n-1}}^{\lambda_{n-1}-\mu_{n-1}} \left\{ \prod_{j=1}^{n} x^{\lambda_j-1} \Phi_2^{(n)} [a, \mu_1, \ldots, \mu_{n-1}, -; c; z_1 x_1, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=1}^{n-1} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \Phi_2^{(n)} [a, \lambda_1, \ldots, \lambda_{n-1}, - ; c; z_1 x_1, \ldots , z_n x_n] ,$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i=1, \ldots , n-1 \quad .$

$$(7.6.23) \quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j-1} {}^{(k)}_{(1)}\Phi_{AC}^{(n)} [a, b; \lambda_1, \ldots, \lambda_n; z_1 x_1, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} \cdot x_j^{\mu_j-1} {}^{(k)}_{(1)}\Phi_{AC}^{(n)} [a, b; \mu_1, \ldots, \mu_n; z_1 x_1, \ldots , z_n x_n] ,$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i=1 \; , \; \ldots \; , \; n-1 \quad .$

$$(7.6.24) \quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=1}^{n} x_j^{\lambda_j-1} \cdot {}^{(k)}_{(2)}\Phi_{AC}^{(n)} \left[a, b_{k+1}, \ldots, b_n; \lambda_1, \ldots, \lambda_n; z_1 x_1, \ldots, z_n x_n \right] \Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} \; x_j^{\mu_j-1} \; {}^{(k)}_{(2)}\Phi_{AC}^{(n)} \left[a, b_{k+1}, \ldots, b_n; \mu_1, \ldots, \mu_n; z_1 x_1, \ldots, z_n x_n \right],$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i=1, \ldots, n \quad .$

$$(7.6.25) \quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \ldots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{i=k+1}^{n} x_j^{\lambda_j-1} \cdot {}^{(k)}_{(2)}\Phi_{AC}^{(n)} \left[a, \mu_{k+1}, \ldots, \mu_n; c_1, \ldots, c_n; z_1, \ldots, z_k, z_{k+1} x_{k+1}, \ldots, z_n x_n \right] \Big\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} \; x_j^{\mu_j-1} \, {}^{(k)}_{(2)}\Phi_{AC}^{(n)} \left[a, \lambda_{k+1}, \ldots, \lambda_n; c_1, \ldots, c_n; z_1, \ldots, z_k, z_{k+1} x_{k+1}, \ldots, z_n x_n \right],$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i=1, \; \ldots \; , n \quad .$

$$(7.6.26) \quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=1}^{n} x_j^{\lambda_j-1} \, {}^{(k)}_{(1)}\Phi_{AD}^{(n)} \left[a, \mu_1, \ldots, \mu_n; c; z_1 x_1, \ldots, z_n x_n \right] \Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} \; x_j^{\mu_j-1} \; {}^{(k)}_{(1)}\Phi_{AD}^{(n)} \left[a, \lambda_1, \ldots, \lambda_n; c; z_1 x_1, \ldots, z_n x_n \right],$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i=1, \; \ldots \; , n \quad .$

$$(7.6.27) \quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=1}^{n} x_j^{\lambda_j-1} \, {}^{(k)}_{(1)}\Phi_{BD}^{(n)} \left[a, \mu_1, \ldots, \mu_n; c; z_1 x_1, \ldots, z_n x_n \right] \Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j -1} \quad {}^{(k)}_{(2)}\Phi^{(n)}_{BD} [a, \lambda_1, \ldots, \lambda_n; c: z_1 x_1, \ldots, z_n x_n],$$

$Re(\lambda_i) > 0 \quad , \quad i = 1 \; , \; \ldots \; , \; n \; .$

$$(7.6.28) \quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \ldots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=k+1}^{n} x_j^{\lambda_j -1} .$$

$$ {}^{(k)}_{(2)}\Phi^{(n)}_{BD} [a, \mu_{k+1}, \ldots, \mu_n; b_1, \ldots, b_n; c: z_1, \ldots, z_k, z_{k+1} x_{k+1}, \ldots, z_n x_n] \Big\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j -1} \quad {}^{(k)}_{(2)}\Phi^{(n)}_{BD} [a, \lambda_{k+1}, \ldots, \lambda_n, b_1, \ldots, b_n; c; z_1, \ldots, z_k, z_{k+1} x_{k+1}, \ldots, z_n x_n],$$

$Re(\lambda_i) > 0 \quad , \quad i=1 \; , \; \ldots \; , \; n \; .$

$$(7.6.29) \quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=1}^{n} x_j^{\lambda_j -1} .$$

$$ {}^{(k)}_{(2)}\Phi^{(n)}_{BD} [a, a_{k+1}, \ldots, a_n, \mu_1, \ldots, \mu_n; c; z_1 x_1, \ldots, z_n x_n] \Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j -1} \quad {}^{(k)}_{(2)}\Phi^{(n)}_{BD} [a, a_{k+1}, \ldots, a_n; \lambda_1, \ldots, \lambda_n; c; z_1 x_1, \ldots, z_n x_n],$$

$Re(\lambda_i) > 0 \quad , \quad i = 1 \; , \; \ldots \; , \; n \; .$

$$(7.6.30) \quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \ldots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=k+1}^{n} x_j^{\lambda_j -1} .$$

$$ {}^{(k)}_{(1)}\Phi^{(n)}_{CD} [a, b; c, \lambda_{k+1}, \ldots, \lambda_n; z_1, \ldots, z_k, z_{k+1} x_{k+1}, \ldots, z_n x_n] \Big\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \; {}^{(k)}_{(1)}\Phi^{(n)}_{CD}[a,b;c,\mu_{k+1},\dots,\mu_n;z_1,\dots,z_k,z_{k+1}x_{k+1},\dots,z_nx_n],$$

$Re(\lambda_j) > 0$, $j = 1$, $k+1$, ... , n .

$$(7.6.31) \quad D_{x_1}^{\lambda_1-\mu_1} \dots D_{x_k}^{\lambda_k-\mu_k} \Big\{ \prod_{j=1}^{k} x_j^{\lambda_j-1} \cdot$$

$${}^{(k)}_{(2)}\Phi^{(n)}_{CD}[a,\mu_1,\dots,\mu_k;c,c_{k+1},\dots,c_n;z_1x_1,\dots,z_kx_k,z_{k+1},\dots,z_n]\Big\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{CD}[a,\lambda_1,\dots,\lambda_k;c,c_{k+1},\dots,c_n;z_1x_1,\dots,z_kx_k,z_{k+1},\dots,z_n],$$

$Re(\lambda_i) > 0$, $j = 1, \dots , k$.

$$(7.6.32) \quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \dots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=k+1}^{n} x_j^{\lambda_j-1} .$$

$${}^{(k)}_{(2)}\Phi^{(n)}_{CD}[a,b_1,\dots,b\,;c,\lambda_{k+1},\dots,\lambda_n;z_1,\dots,z_k,z_{k+1}x_{k+1},\dots,z_nx_n]\Big\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{CD}[a,b_1,\dots,b_k;c,\mu_{k+1},\dots,\mu_n;z_1,\dots,z_k,z_{k+1}x_{k+1},\dots,z_nx_n],$$

$Re(\lambda_i) > 0$, $i = k+1$, ... , n .

$$(7.6.33) \quad D_{x_1}^{\lambda_1-\mu_1} \dots D_{x_k}^{\lambda_k-\mu_k} \Big\{ \prod_{j=1}^{k} x_j^{\lambda_j-1} .$$

$${}^{(k)}_{(3)}\Phi^{(n)}_{CD}[b,\mu_1,\dots,\mu_k;c,c_{k+1},\dots,c_n;z_1x_1,\dots,z_kx_k,z_{k+1}x_{k+1},\dots,z_nx_n]\Big\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \; {}^{(k)}_{(3)}\Phi^{(n)}_{CD}[b,\lambda_1,\dots,\lambda_k;c,c_{k+1},\dots,c_n;z_1x_1,\dots,z_kx_k,z_{k+1}x_{k+1},\dots,z_nx_n],$$

$\mathrm{Re}(\lambda_j) > 0 \quad , \quad j = 1, \ldots, k \ .$

$$(7.6.34) \quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \cdots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=k+1}^{n} x_j^{\lambda_j-1}$$

$$ {}_{(3)}^{(k)}\Phi_{CD}^{(n)} [b, b_1, \ldots, b_k; c, \lambda_{k+1}, \ldots, \lambda_n; z_1, \ldots, z_k, z_{k+1}x_{k+1}, \ldots, z_n x_n] \Big\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \; {}_{(3)}^{(k)}\Phi_{CD}^{(n)} [b, b_1, \ldots, b_k; c, \mu_{k+1}, \ldots, \mu_n; z_1, \ldots, z_k, z_{k+1}x_{k+1}, \ldots, z_n x_n] ,$$

$\mathrm{Re}(\lambda_j) > 0 \quad , \quad j = k+1, \ldots, n \ .$

$$(7.6.35) \quad D_{x_1}^{\lambda_1-\mu_1} \cdots D_{x_k}^{\lambda_k-\mu_k} \Big\{ \prod_{j=1}^{k} x_j^{\lambda_j-1} \ .$$

$$ {}_{(4)}^{(k)}\Phi_{CD}^{(n)} [a, b, \mu_1, \ldots, \mu_k; c; z_1x_1, \ldots, z_kx_k, z_{k+1}, \ldots, z_n] \Big\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \; {}_{(4)}^{(k)}\Phi_{CD}^{(n)} [a, b, \lambda_1, \ldots, \lambda_k; c; z_1x_1, \ldots, z_kx_k, z_{k+1}, \ldots, z_n] ,$$

$\mathrm{Re}(\lambda_j) > 0 \quad , \quad j = 1, \ldots, k \ .$

$$(7.6.36) \quad D_{x_1}^{\lambda_1-\mu_1} \cdots D_{x_k}^{\lambda_k-\mu_k} . D_{y_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \cdots D_{y_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=1}^{k} x_j^{\lambda_j-1} . \prod_{j=k+1}^{n} x_j^{\lambda_j-1} .$$

$$ {}_{(2)}^{(k)}\Phi_{CD}^{(n)} [a, \mu_1, \ldots, \mu_k; c, \lambda_{k+1}, \ldots, \lambda_n; z_1x_1, \ldots, z_kx_k, z_{k+1}y_{k+1}, \ldots, z_n y_n] \Big\}$$

$$= \prod_{i=1}^{k} \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_i)\,\Gamma(\lambda_j)}{\Gamma(\mu_i)\,\Gamma(\mu_j)} x_i^{\mu_i-1} . y_j^{\mu_j-1} .$$

$$ {}_{(2)}^{(k)}\Phi_{CD}^{(n)} [a, \lambda_1, \ldots, \lambda_k; c, \mu_{k+1}, \ldots, \mu_n; z_1x_1, \ldots, z_kx_k, z_{k+1}y_{k+1}, \ldots, z_n y_n] ,$$

$\mathrm{Re}(\lambda_i) > 0 \quad , \quad i = 1, \ldots, n \ ,$

$$(7.6.37)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_k}^{\lambda_k-\mu_k}\,.D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}}\cdots D_{x_n}^{\lambda_n-\mu_n}\Big\{\prod_{i=1}^{k} x_i^{\lambda_i-1}\cdot\prod_{j=k+1}^{n} x_j^{\lambda_j-1}$$

$${}^{(k)}_{(3)}\Phi^{(n)}_{CD}\,[\,b,\mu_1,\ldots,\mu_k;c,\lambda_{k+1},\ldots,\lambda_n;z_1x_1,\ldots,z_kx_k,z_{k+1}x_{k+1},\ldots,z_nx_n\,]$$

$$=\prod_{i=1}^{k}\cdot\prod_{j=k+1}^{n}\frac{\Gamma(\lambda_i)\ \Gamma(\lambda_j)}{\Gamma(\mu_i)\ \Gamma(\mu_j)}\,x_i^{\mu_i-1}\cdot y_j^{\mu_j-1}\,.$$

$${}^{(k)}_{(3)}\Phi^{(n)}_{CD}\,[\,b,\lambda_1,\ldots,\lambda_k;c,\mu_{k+1},\ldots,\mu_n;z_1x_1,\ldots,z_kx_k,z_{k+1}y_{k+1},\ldots,z_ny_n\,],$$

$\mathrm{Re}(\lambda_i)>0 \quad , \quad i=1,\ldots,n\ .$

$$(7.6.38)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_n}^{\lambda_n-\mu_n}\Big\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\,.$$

$${}^{(k)}_{(5)}\Phi^{(n)}_{CD}\,[\,a,b,\mu_1,\ldots,\mu_k;\lambda_{k+1},\ldots,\lambda_n;z_1x_1,\ldots,z_nx_n\,]\Big\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\ {}^{(k)}_{(5)}\Phi^{(n)}_{CD}\,[\,a,b,\lambda_1,\ldots,\lambda_k;\mu_{k+1},\ldots,\mu_n;z_1x_1,\ldots,z_nx_n\,],$$

$\mathrm{Re}(\lambda_i)>0 \quad , \quad j=1,\ldots,n\ .$

$$(7.6.39)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_n}^{\lambda_n-\mu_n}\Big\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\,.$$

$${}^{(k)}_{(6)}\Phi^{(n)}_{CD}\,[\,a,\mu_1,\ldots,\mu_k;\lambda_{k+1},\ldots,\lambda_n;z_1x_1,\ldots,z_nx_n\,]\Big\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\ {}^{(k)}_{(6)}\Phi^{(n)}_{CD}\,[\,a,\lambda_1,\ldots,\lambda_k;\mu_{k+1},\ldots,\mu_n;z_1x_1,\ldots,z_nx_n\,],$$

$\mathrm{Re}(\lambda_i)>0 \quad , \quad i=1,\ldots,n\ .$

$$(7.6.40)\quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \cdots D_{x_n}^{\lambda_n-\mu_n}\Big\{\prod_{j=k+1}^{n} x_j^{\lambda_j-1}$$

$$ {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,\mu_{k+1},\dots,\mu_n,b_1,\dots,b_k;c;z_1,\dots,z_k,z_{k+1}x_{k+1},\dots,z_nx_n\right]\Big\}$$

$$=\prod_{j=k+1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}x_j^{\mu_j-1}\;{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,\lambda_{k+1},\dots,\lambda_n,b_1,\dots,b_k;c;z_1,\dots,z_k,z_{k+1}x_{k+1},\dots,z_nx_n\right],$$

$Re(\lambda_j)>0$, $j=k+1,\dots,n$.

$$(7.6.41)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_n}^{\lambda_n-\mu_n}\Big\{\prod_{j=1}^{k} x_j^{\lambda_j-1}\cdot$$

$$ {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,a_{k+1},\dots,a_n,\mu_1,\dots,\mu_k;c;z_1x_1,\dots,z_kx_k,x_{k+1},\dots,x_n\right]\Big\}$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}x_j^{\mu_j-1}\;{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,a_{k+1},\dots,a_n,\lambda_1,\dots,\lambda_k;c;z_1x_1,\dots,z_kx_k,z_{k+1},\dots,z_n\right],$$

$Re(\lambda_j)>0$, $j=1,\dots,k$.

$$(7.6.42)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_n}^{\lambda_n-\mu_n}\Big\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\cdot$$

$$ {}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,\mu_{k+1},\dots,\mu_n,\mu_1,\dots,\mu_k;c;z_1x_1,\dots,z_nx_n\right]\Big\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}x_j^{\mu_j-1}\;{}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,\lambda_{k+1},\dots,\lambda_n,\lambda_1,\dots,\lambda_k;z_1x_1,\dots,z_nx_n\right],$$

$Re(\lambda_j)>0$, $j=1,\dots,n$.

$$(7.6.43)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\cdot {}^{(k)}_{(1)}\phi_D^{(n)}[a,\mu_1,\ldots,\mu_n;c;\ z_1x_1,\ldots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\ {}^{(k)}_{(1)}\phi_D^{(n)}[a,\lambda_1,\ldots,\lambda_n;c;z_1x_1,\ldots,z_nx_n],$$

$Re(\lambda_j)>0$, $j=1,\ldots,n$.

$$(7.6.44)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\ {}^{(k)}_{(2)}\phi_D^{(n)}[a,\mu_1,\ldots,\mu_n;c;z_1x_1,\ldots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\ {}^{(k)}_{(2)}\phi_D^{(n)}[a,\lambda_1,\ldots,\lambda_n;c;\ z_1x_1,\ \ldots,\ z_nx_n],$$

$Re(\lambda_j)>0$, $j=1,\ldots,n$.

$$(7.6.45)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\ {}^{(k)}_{(1)}\phi_C^{(n)}[a,b;\lambda_1,\ldots,\lambda_n;z_1x_1,\ldots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\ {}^{(k)}_{(1)}\phi_C^{(n)}[a,b;\mu_1,\ldots,\mu_n;z_1x_1,\ldots,z_nx_n],$$

$Re(\lambda_j)>0$, $j=1,\ldots,n$.

$$(7.6.46)\quad D_{x_1}^{\lambda_1-\mu_1}\cdots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\cdot {}^{(k)}_{(2)}\phi_{AD}^{(n)}[a,\mu_1,\ldots,\mu_n;c_{k+1},\ldots,c_n;z_1x_1,\ldots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{(\lambda_j)}{(\mu_j)}\,x_j^{\mu_j-1}\ {}^{(k)}_{(2)}\phi_{AD}^{(n)}[a,\lambda_1,\ldots,\lambda_n;c_{k+1},\ldots,c_n;\ z_1x_1,\ldots,z_nx_n],$$

$Re(\lambda_j)>0$, $j=1,\ldots,n$.

(7.6.47) $D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \cdots D_{x_n}^{\lambda_n-\mu_n} \Big\{ \prod_{j=k+1}^{n} x_j^{\lambda_j-1} \cdot$

$$ {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\big[a,b_1,\ldots,b_n;\lambda_{k+1},\ldots,\lambda_n;z_1,\ldots,z_k,z_{k+1}x_{k+1},\ldots,z_nx_n\big]\Big\} $$

$$ \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\big[a,b_1,\ldots,b_n;\mu_{k+1},\ldots,\mu_n;z_1,\ldots,z_k,z_{k+1}x_{k+1}, \ldots,z_nx_n\big], $$

$Re(\lambda_j) > 0$, $j=k+1, \ldots, n$.

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APPLICATIONS OF MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTVA AND DAOUST IN STATISTICS

CHAPTER VIII

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APPLICATIONS OF MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST IN STATISTICS

8.1 Introduction Different distributions have been discussed by various authors Block and Rao [1], Carlsson [2], Daley [4], Datt [5], Kabe [7], Kaufman, Mathai and Saxena [8], Kendall [9], Khatri and Pillai [10,11], Khatri and Srivastava [12]. Littler and Fackerell [14], Lukacs and Naha [15]. Lukacs [16], Mathai ([17] to [28]) Mathai and Rathie ([29] to [34]), Mathai and Saxena ([35] to [41]), Miller [42] , Pillai, Al-Ani and Jouris [43], Pillai and Jouris [44], Pillai and Nagarsenker [45], Robbins and Pitman [46], Strawderman [51], Thaung [52] and Wilks [53]. Srivastava and Singhal [49] studied many of the classical statistical distributions, which were associated with the beta and gamma distributions. Further Exton [6] discussed generalized beta and gamma distributions with other special multivariate distributions like Dirichlet distributions and multivariate normal distributions.

He also discussed the expectations of some functions involving Lauricella's multiple hypergeometric functions [13].

In the present chapter, we extend the above work and establish some probability density functions associated with the multivariate beta and gamma distributions and make their applications to obtain some expectations involving the most generalized multiple hypergeometric function of Srivastava and Daoust [47] (see also Srivastava and Manocha [50], p.64) Finally, we also derive the moments for these multivariate beta and gamma distributions and discuss their special cases.

8.2 FORMULAE REQUIRED

For ready stock, in this section , we write the following results which will be used in our investigations:

The Liouville's theorem (Also see Chandel [3, p.83(3.1)])

$$(8.2.1) \quad \int_0^{\infty} \cdots \int_0^{\infty} f(x_1+\ldots+x_n) \; x_1^{\mu_1-1} \cdots x_n^{\mu_n-1} \, dx_1 \ldots dx_n$$

$$= \frac{\Gamma(\mu_1) \ldots \Gamma(\mu_n)}{\Gamma(\mu_1+\ldots+\mu_n)} \int_0^{\infty} f(t) \; t^{\mu_1+\ldots+\mu_n-1} \, dt ,$$

provided that $(x_1,+\ldots+x_n) \geqslant 0$, for all positive values of $x_1,\ldots,x_n$.

Euler's definition for gamma function

$$(8.2.2) \quad \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \ , \quad Re(z) > 0 \ .$$

The definition of beta function (see, Srivastava and Manocha [50, p.26 eq. (46)]

$$(8.2.3) \quad B(\alpha,\beta) = \int_0^{\infty} \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du \ , \quad Re(\alpha) > 0 \ , \quad Re(\beta) > 0 \ .$$

8.3 Multivariate Gamma Distribution

Consider the function

$$(8.3.1) \quad f(x_1,\ldots,x_n) = \frac{\Gamma(\mu_1+\ldots+\mu_n) \ \lambda^{\mu+\mu_1+\ldots+\mu_n}}{\Gamma(\mu_1)\ldots\Gamma(\mu_n) \ \Gamma(\mu+\mu_1+\ldots+\mu_n)} e^{-(x_1+\ldots+x_n)\lambda}$$

$$(x_1+\ldots+x_n)^{\mu} x_1^{\mu_1-1} \ldots x_n^{\mu_n-1} \ ,$$

provided that $Re(\lambda) > 0$, $(x_1+\ldots+x_n) \geqslant 0$, $x_i \geqslant 0$, $Re(\mu_i) > 0$, $i = 1, \ldots, n$, and $f(x_1,\ldots,x_n) = 0$ else where .

Making an appeal to (8.2.1) and (8.2.2) , the value of multiple integral of $f(x_1,\ldots,x_n)$ over the region defined above in (8.3.1) becomes unity. Hence $f(x_1,\ldots,x_n)$ is a probability density function for multivariate gamma distribution .

8.4 Expectation Associated with Multivariate Gamma Distribution

The expectation value of the function $g(x_1,\dots,x_n)$ is defined as

$$(8.4.1) \quad \langle g(x_1,\dots,x_n)\rangle = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\dots,x_n)\, g(x_1,\dots,x_n)\, dx_1\dots dx_n .$$

Corresponding to density function $f(x_1,\dots,x_n)$ defined by (8.3.1), consider the function

$$(8.4.2) \quad g_1(x_1,\dots,x_n) = F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:\\ [(c):\psi',\dots,\psi^{(n)}]:\end{matrix}\ \begin{matrix}[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1(x_1+\dots+x_n)^{\nu_1},\dots,z_n(x_1+\dots+x_n)^{\nu_n}\right)$$

where $F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}$ is most generalized multiple hypergeometric

function of Srivastava and Daoust [47] (Also see Srivastava and Manocha [50,(18),(19),(20),p.64]).

Now making an appeal to (8.2.1) and (8.2.2), the expectation of $g_1(x_1,..,x_n)$ having density function $f(x_1,..,x_n)$ is given by

$$(8.4.3)\quad \langle g_1(x_1,...,x_n)\rangle = F^{A+1\,:\,B';\ldots;B^{(n)}}_{C\,:\,D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: [\rho+\mu_1+\ldots+\mu_n:\nu_1,\ldots,\nu_n]: [(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]: [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\ \frac{z_1}{\lambda_1^{\nu_1}},\ldots,\frac{z_n}{\lambda_n^{\nu_n}}\right),$$

provided that

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\nu_i>0,$$

$i=1,\ldots,n$.

Corresponding to density function $f(x_1,..,x_n)$ defined by (8.3.1), if we consider the function

$$(8.4.4) \quad g_2(x_1,\dots,x_n) = F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]: [(b'):\Phi'];\dots;[(b^{(n)}):\Phi^{(n)}]; \\ [(c):\Psi',\dots,\Psi^{(n)}]: [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1 x_1^{\alpha_1}(x_1+\dots+x_n)^{\nu_1},\dots,z_n x_n^{\alpha_n}(x_1+\dots+x_n)^{\nu_n}\right)$$

Then the expectation of g_2 is given by

$$(8.4.5) \quad \langle g_2(x_1,\dots,x_n)\rangle = F^{A+1:B'+1;\dots;B^{(n)}+1}_{C+1:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]: [\mu+\mu_1+\dots+\mu_n;\alpha_1+\nu_1,\dots,\alpha_n+\nu_n]; [(b'):\Phi'],[\mu_1:\alpha_1];\dots;[(b^{(n)}):\Phi^{(n)}],[(\mu_n):\alpha_n]; \\ [(c):\Psi',\dots,\Psi^{(n)}]: [\mu_1+\dots+\mu_n;\alpha_1+\dots+\alpha_n]; [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; \frac{z_1}{\lambda^{\alpha_1+\nu_1}},\dots,\frac{z_n}{\lambda^{\alpha_n+\nu_n}}\right),$$

valid if

$$1 + \sum_{j=1}^{C} \gamma_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} - \nu_i - \alpha_i > 0 ,$$

$i = 1, \ldots, n$.

8.5 The Multivariate Beta Distribution

Consider the function $F(x_1, \ldots, x_n)$ defined by

$$(8.5.1) \quad F(x_1, \ldots, x_n) = \frac{\Gamma(\mu_1 + \ldots + \mu_n)\, \Gamma(\alpha + \lambda + \mu_1 + \ldots + \mu_n)\, (x_1 + \ldots + x_n)^{\alpha}}{\Gamma(\mu_1) \ldots \Gamma(\mu_n)\, \Gamma(\lambda)\, \Gamma(\alpha + \mu_1 + \ldots + \mu_n)\, (1 + x_1 + \ldots + x_n)^{\alpha + \lambda + \mu_1 + \ldots + \mu_n}} \cdot x_1^{\mu_1 - 1} \ldots x_n^{\mu_n - 1} ,$$

$Re(\alpha) > 0$, $Re(\mu_i) > 0$, $x_i > 0$, $i = 1, \ldots, n$ and $F(x_1, \ldots, x_n) = 0$ else where .

Now making an appeal to (8.2.1) and (8.2.3), the value of multiple integral of $F(x_1, \ldots, x_n)$ over the region defined above in (8.5.1), becomes unity , Hence $F(x_1, \ldots, x_n)$ is probability density function for multivariate beta distribution .

8.6 Expectation Associated with Multivariate Beta Distribution

Corresponding to density function $F(x_1,\ldots,x_n)$ defined by (8.5.1)

Consider the function

$$(8.6.1)\quad G_1(x_1,\ldots,x_n) = F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: \\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\quad \frac{z_1 x_1^{\xi_1}(x_1+\ldots+x_n)^{\eta_1}}{(1+x_1+\ldots+x_n)^{\xi_1+\eta_1}},\ldots,\frac{z_n x_n^{\xi_n}(x_1+\ldots+x_n)^{\eta_n}}{(1+x_1+\ldots+x_n)^{\xi_n+\eta_n}}\right)$$

Then the expectation of $G(x_1,\ldots,x_n)$ is given by

$$(8.6.2)\ \langle G(x_1,\ldots,x_n)\rangle = F^{A+1:\,B'+1;\ldots;B^{(n)}+1}_{C+2:D';\ldots;\,D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], \\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$[\alpha+\mu_1+\ldots+\mu_n:\eta_1+\xi_1,\ldots,\eta_n+\xi_n]:\ [(b'):\phi'],\ [\mu_1:\xi_1];\ldots;\ [(b^{(n)}):\phi^{(n)}],$$

$$[\mu_1+\ldots+\mu_n:\xi_1,\ldots,\xi_n],\ [\alpha+\lambda+\mu_1+\ldots+\mu_n:\eta_1+\xi_1,\ldots,\eta_n+\xi_n]:\ [(d'):\delta'];$$

$$\left.\begin{matrix}[\mu_n:\xi_n]; \\ \ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\quad z_1,\ldots,z_n\right),$$

where

$$1 + \sum_{j=1}^{C} \tau_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} + \gamma_i - \xi_i > 0 \; ,$$

$i=1, \ldots, n$.

8.7 Moment Generating Function (M.G.F.) For Gamma Distribution

The m.g.f. is defined as

$$(8.7.1) \quad M(t_1, \ldots, t_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{x_1 t_1 + \ldots + x_n t_n} \, f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n \; ,$$

provided that the integral is a function of the parameters $t_1, \ldots, t_n$ only .

Thus m.g.f. for multivariate gamma distribution (8.3.1) is given by

$$(8.7.2) \quad M(t_1, \ldots, t_n) = \frac{\Gamma(\mu_1 + \ldots + \mu_n) \lambda^{\mu + \mu_1 + \ldots + \mu_n}}{\Gamma(\mu_1) \ldots \Gamma(\mu_n) \, \Gamma(\mu + \mu_1 + \ldots + \mu_n)} \; .$$

$$\int_0^{\infty} \cdots \int_0^{\infty} e^{x_1 t_1 + \ldots + x_n t_n} \, . \, e^{-(x_1 + \ldots + x_n)} \, (x_1 + \ldots + x_n) \, x_1^{\mu_1 - 1} \ldots x_n^{\mu_n - 1} \, . \, dx_1 \ldots dx_n$$

Now making an appeal to (8.2.1) and the result due to Srivastava [48, p.4 (12)]

$$\sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1+\ldots+m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} = \sum_{M=0}^{\infty} f(M)\frac{(x_1+\ldots+x_n)^M}{M!}, \quad n \geq 1$$

We finally derive

(8.7.3) $$M(t_1,\ldots,t_n) = F_D^{(n)}\left[\mu+\mu_1+\ldots+\mu_n, \mu_1,\ldots,\mu_n; \mu_1+\ldots+\mu_n; \frac{t_1}{\lambda}\cdots\frac{t_n}{\lambda}\right],$$

where $F_D^{(n)}$ is Lauricella's fourth multiple hypergeometric **function** of several variables [13].

As a special case for $\mu=0$, (8.7.3) gives

(8.7.4) $$M(t_1,\ldots,t_n) = \prod_{i=1}^{n}\left(1-\frac{t_i}{\lambda}\right)^{-\mu_i},$$

8.8 Moments for Gamma Distribution

The moment $\mu'_{r_1,\ldots,r_n}$ for gamma distribution about $(0,0,\ldots,0)$ of order $r_1,\ldots,r_n$ is defined as the coefficient of $\frac{t_1^{r_1}}{r_1!}\cdots\frac{t_n^{r_n}}{r_n!}$ in $M(t_1,\ldots,t_n)$ when it is expanded in powers of $t_1,\ldots,t_n$.

Thus an appeal to (8.2.1) gives

$$(8.8.1)\quad \mu'_{r_1,\ldots,r_n} = \frac{(\mu_1)_{r_1}\ldots(\mu_n)_{r_n}\,(\mu+\mu_1+\ldots+\mu_n)_{r_1+\ldots+r_n}}{(\mu_1+\ldots+\mu_n)_{r_1+\ldots+r_n}\cdot\lambda^{r_1+\ldots+r_n}}$$

8.9 Moment For Beta Distribution

The moment $\mu'_{r_1,\ldots,r_n}$ of density function $F(x_1,\ldots,x_n)$ about $(0,\ldots,0)$ for beta distribution is defined as

$$(8.9.1)\quad \mu'_{r_1,\ldots,r_n} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{r_1}\ldots x_n^{r_n}\,F(x_1,\ldots,x_n)\,dx_1\ldots dx_n\,.$$

Now substituting the value of $F(x_1,..,x_n)$ from (8.5.1) in (8.9.1) and making an appeal to (8.2.1) and (8.2.3), we finally derive

$$(8.9.2)\quad \mu'_{r_1,\ldots,r_n} = \frac{\Gamma[\alpha+\lambda-(r_1+..+r_n)]}{\Gamma(\lambda)\,\Gamma(\alpha+\mu_1+\ldots+\mu_n)}\cdot\frac{(\mu_1)_{r_1}\ldots(\mu_n)_{r_n}}{(\mu_1+\ldots+\mu_n)_{r_1+..+r_n}}\,.$$

8.10 Special Cases

For $n=1$, from (8.7.3), we derive the following m.g.f. for gamma distribution :

$$(8.10.1) \qquad M(t_1) = \left(1 - \frac{t_1}{\lambda}\right)^{-\mu+\mu_1}$$

For $n=1$, from (8.8.1), we obtain following moment of r_1th order about origin for gamma distribution :

$$(8.10.2) \qquad \mu'_{r_1} = \frac{(\mu+\mu_1)_{r_1}}{\lambda^{r_1}} .$$

Also for $n=1$, (8.9.2) gives following moment of r_1th order for beta distribution

$$(8.10.3) \qquad \mu'_{r_1} = \frac{\Gamma(\alpha+\lambda-r_1)}{\Gamma(\lambda)\,\Gamma(\alpha+\mu_1)} .$$

Now for special interest, we shall discuss the applications of other multiple hypergeometric functions of several variables in Statistics in next chapter .

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APPLICATIONS OF OTHER MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES IN STATISTICS

CHAPTER IX

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APPLICATIONS OF OTHER MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES IN STATISTICS

9.1 Introduction . Exton [7,p.222] studied many special multivariate distributions having expectations in terms of Lauricella's multiple hypergeometric functions [10]. In the previous chapter VIII , we obtained some density functions associated with the multivariate gamma and beta distributions and made their applications to derive the expectations involving multiple hypergeometric functions of Srivastava and Daoust [11] .

Motivated by the above work for special interest, here in the present chapter, we discuss some multivariate beta and gamma distributions and make their applications to derive some expectations of different functions in terms of multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_{D}$, ${}^{(k)}_{(2)}E^{(n)}_{D}$ of Exton [5,7] , ${}^{(k)}_{(1)}E^{(n)}_{C}$ of Chandel [1] , ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$ of Chandel and Gupta [3] and ${}^{(k)}F^{(n)}_{CD}$ of Karlsson [9] and their confluent forms introduced by Chandel and Gupta [3] (Also see Gupta [8] and Chandel and Vishwakarma [4]

9.2 Expectations of different functions related to multivariate beta distributions

The expectation for the function $g(x_1,\ldots,x_n)$ having multivariate density function $f(x_1,\ldots,x_n)$ is defined as

$$(9.2.1)\quad \langle g(x_1,\ldots,x_n)\rangle = \int_{-\infty}^{\infty}\cdot(n)\cdot\int_{-\infty}^{\infty} f(x_1,\ldots,x_n)\cdot g(x_1,\ldots,x_n)\,.dx_1\ldots dx_n$$

We consider the density function

$$(9.2.2)\quad f_1(x_1,\ldots,x_n) = x_1^{b_1-1}\ldots x_n^{b_n-1}\,(1-x_1-\ldots-x_k)^{c-b_1-\ldots-b_k-1}\,.$$
$$(1-x_{k+1}-\ldots-x_n)^{c'-b_{k+1}-\ldots-b_n-1}\;\frac{\Gamma(c)\;\Gamma(c')}{\Gamma(b_1)\ldots\Gamma(b_n)\Gamma(c-b_1-\ldots-b_k)\Gamma(c'-b_{k+1}-\ldots-b_n)}$$

provided that $0\le x_1,\ldots,0\le x_n$, $x_1+\ldots+x_k\le 1$, $x_{k+1}+\ldots+x_n\le 1$ and real parts of c, c', $c-b_1-\ldots-b_k$ and $c'-b_{k+1}-\ldots-b_n$ are positive and $f_1=0$ elsewhere;
and another density function

$$(9.2.3)\quad f_2(x_1,\ldots,x_n) = x_1^{b_1-1}\ldots x_n^{b_n-1}\;(1-x_1-\ldots-x_k)^{c-b_1-\ldots-b_k-1}$$
$$(1-x_{k+1})^{c_{k+1}-b_{k+1}-1}\ldots(1-x_n)^{c_n-b_n-1}\,\frac{\Gamma(c)\;\Gamma(c_{k+1})\ldots\Gamma(c_n)}{\Gamma(b_1)\ldots\Gamma(b_n)\Gamma(c-b_1-\ldots-b_k)}\,.$$
$$\frac{1}{\Gamma(c_{k+1}-b_{k+1})\ldots\Gamma(c_n-b_n)}$$

where $x_1+\ldots+x_k \leq 1$, $0 \leq x_r \leq 1$, $r = 1, \ldots, n$.
$Re(c) > Re(b_1+\ldots+b_k)$, $Re(b_j) > 0$, $j = 1, \ldots, k$,
$Re(c_i) > Re(b_i) > 0$, $i \in \{k+1, \ldots, n\}$, and $f_2 = 0$, else where .

Corresponding to density function f_1 defined by (9.2.2), consider the function

$$(9.2.4) \qquad g_1(x_1,\ldots,x_n) = (1-x_1\alpha_1-\ldots-x_n\alpha_n)^{-a} .$$

Now putting the value of f_1 and g_1 from (9.2.2) and (9.2.4) respectively in (9.2.1) and making an appeal to the result due to Exton [7, p. 93 , (3.4.2.4)] we obtain the expectation for $g_1(x_1,\ldots,x_n)$ having density function $f_1(x_1,\ldots,x_n)$

$$(9.2.5) \quad < g_1(x_1,\ldots,x_n) > = {}^{(k)}_{(1)}E^{(n)}_{D} [a, b_1,\ldots,b_n; c, c' : x_1,\ldots,x_n]$$

where ${}^{(k)}_{(1)}E^{(n)}_{D}$ is multiple hypergeometric function related to Lauricella's $F^{(n)}_{D}$ introduced by Exton [5] .

Further putting the values of f_2 and g_1 from (9.2.3) and (9.2.4) respectively in (9.2.1) and making an appeal to the result due to Karlsson [9, (3.2)] , we derive the following expectation value of g_1 corresponding to density function f_2 ;

(9.2.6) $\langle g_1(x_1,\ldots,x_n)\rangle = {}^{(k)}F^{(n)}_{AD}[a, b_1,\ldots,b_n; c, c_{k+1},\ldots,c_n; \alpha_1,\ldots,\alpha_n]$

where ${}^{(k)}F^{(n)}_{AD}$ is one of the intermediate Lauricella's multiple hypergeometric function introduced by Chandel and Gupta [3].

We now consider the density function

(9.2.7) $$f_3(x_1,\ldots,x_n) = x_1^{b_1-1}\ldots x_n^{b_n-1}(1-x_1-\ldots-x_n)^{c-b_1-\ldots-b_n-1}\cdot\frac{\Gamma(c)}{\Gamma(b_1)\ldots\Gamma(b_n)\;\Gamma(c-b_1-\ldots-b_n)}$$

provided that $0\leq x_1,\ldots, 0\leq x_n$, $x_1+\ldots+x_n\leq 1$ and all the real parts of c and $c-b_1-\ldots-b_n$ are positive, and $f_3=0$, else where.

Consider

(9.2.8) $$g_2(x_1,\ldots,x_n) = (1-\alpha_1x_1-\ldots-\alpha_kx_k)^{-a}(1-\alpha_{k+1}x_{k+1}-\ldots-\alpha_nx_n)^{-a'}$$

Then putting these values of f_3 and g_2 from (9.2.7) and (9.2.8) respectively in (9.2.1) and making an appeal to the result due to Exton [7, p. 93, (3.4.2.5)], we derive the following expectation of g_2 corresponding to density function f_3

$$(9.2.9) \quad \langle g_2(x_1,\ldots,x_n)\rangle = {}^{(k)}E^{(n)}_{(2)\,D}\left[a,a',b_1,\ldots,b_n;c;\ \alpha_1,\ldots,\alpha_n\right]$$

where ${}^{(k)}E^{(n)}_{(2)\,D}$ is another multiple hypergeometric function related to Lauricella's $F^{(n)}_D$, introduced by Exton [5]

Consider the function

$$(9.2.10) \quad g_3(x_1,\ldots,x_n) = (1-\alpha_1x_1-\ldots-\alpha_kx_k)^{-a}\,(1-\alpha_{k+1}x_{k+1})^{-a_{k+1}}\ \ldots\ (1-\alpha_nx_n)^{-a_n} .$$

Now putting the values of f_3 and g_3 from (9.2.7) and (9.2.10) respectively in (9.2.1) and making an appeal to the result due to Karlsson [9, (3.1)], we get the expectation values of g_3 corresponding to the function f_3

$$(9.2.11) \quad \langle g_3(x_1,\ldots,x_n)\rangle = {}^{(k)}F^{(n)}_{BD}\left[a,a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;\ \alpha_1,\ldots,\alpha_n\right]$$

where ${}^{(k)}F^{(n)}_{BD}$ is one of intermediate Lauricella's multiple hypergeometric functions due to Chandel and Gupta [3].

9.3 Expectations of different multiple hypergeometric functions related to Multivariate gamma distribution

In this section, we discuss some density functions

associated with gamma distributions and derive some expectations in involving multiple hypergeometric function.

We consider the density function

$$(9.3.1) \qquad f(x,y,z) = \frac{1}{\Gamma(a)\ \Gamma(a')\ \Gamma(b)}\, e^{-x-y-z}\, x^{a-1}\, y^{a'-1}\, z^{b-1}$$

provided that $0 \leq x,y,z < \infty$, $Re(a)$, $Re(a')>0$, $Re(b)>0$,

and $f(x,y,z) = 0$, else where .

Consider another function

$$(9.3.2) \qquad g(x,y,z) = {}_0F_1[-;c_1;x_1xz] \ldots {}_0F_1[-;c_k;x_kxz]\cdot {}_0F_1[-;c_{k+1};x_{k+1}yz] \ldots {}_0F_1[-;c_n;x_nyz] .$$

Now making an appeal to the result due to Exton [7, p. 96(3.4.(4.6)], we derive the following expectation value of $g(x,y,z)$ corresponding to the function $f(x,y,z)$

$$(9.3.3) \qquad \langle g(x,y,z)\rangle = {}^{(k)}_{(1)}E^{(n)}_{C}\,(a,a',b;c_1,\ldots,c_n;x_1,\ldots,x_n] ,$$

where ${}^{(k)}_{(1)}E^{(n)}_{C}$ is multiple hypergeometric function related to Lauricella's $F^{(n)}_{C}$ introduced by Chandel [1] .

Now consider the density function

$$(9.3.4) \quad f(y,x_1,\ldots,x_n) = \frac{1}{\Gamma(a)\,\Gamma(b_1)\ldots\Gamma(b_n)}\, e^{-y-x_1-\ldots-x_n}\, y^{a-1}\, .\, x_1^{b_1-1}\ldots x_n^{b_n-1}$$

provided that $0 \leq y,x_1,\ldots,x_n < \infty$ and $Re(a) > 0$, $Re(b_i) > 0$, $i=1,\ldots,n$ and $f = 0$ else where .

Take

$$(9.3.5) \quad g(y,x_1,\ldots,x_n) = {}_0F_1 [-;c;\alpha_1 x_1 y + \ldots + \alpha_k x_k y] \, . \, {}_0F_1 [-;c;\alpha_{k+1} y + \ldots + \alpha_n x_n y] \, .$$

Now putting the values of f and g from (9.3.4) and (9.3.5) respectively in (9.2.1) and making an appeal to the result due to Chandel and Gupta [2, (2.2)] , we obtain the following expectation :

$$(9.3.6) \quad \langle g(y,x_1,\ldots,x_n)\rangle = {}^{(k)}_{(1)}E^{(n)}_{D} [a,b_1,\ldots,b_n;c,c';\alpha_1, \ldots ,\alpha_n] \, .$$

Now we consider the density function

$$(9.3.7) \quad f(z,y,x_1,\ldots,x_n) = \frac{1}{\Gamma(a)\,\Gamma(a')\,\Gamma(b_1)\ldots\Gamma(b_n)}\, e^{-z-y-x_1-\ldots-x_n} \, . \, z^{a-1}\, y^{a'-1}\, x_1^{b_1-1}\ldots x_n^{b_n-1} ,$$

where $0 \leq z,y,x_1,\ldots,x_n \leq \infty$ and all $Re(a)$, $Re(a')$, $Re(b_1)$, ... , $Re(b_n) > 0$; and $f = 0$ else where .

Consider the function

$$(9.3.8) \quad g(z,y,x_1,\ldots,x_n) = {}_0F_1\left[-;c;\ (\alpha_1x_1+\ldots+\alpha_kx_k)z + (\alpha_{k+1}x_{k+1} + \ldots + \alpha_n \cdot x_n)y\right].$$

Now putting the values of f and g from (9.3.7) and (9.3.8) respectively in (9.2.1) and then making an appeal to the result due to Chandel and Gupta [2,(2.3)], we get the following expectation of g corresponding to the function f :

$$(9.3.9) \quad \langle g(z,y,x_1,\ldots,x_n)\rangle = {}^{(k)}_{(2)}E^{(n)}_{D}\left[a,a',b_1,\ldots,b_n;c;\ \alpha_1,\ldots,\alpha_n\right]$$

Consider the density function

$$(9.3.10) \quad f(x,x_{k+1},\ldots,x_n,y_1,\ldots,y_n)$$

$$= \frac{1}{\Gamma(a)\Gamma(a_{k+1})\ldots\Gamma(a_n)\ \Gamma(b_1)\ \ldots\ \Gamma(b_n)}\, x^{a-1}\, x_{k+1}^{a_{k+1}-1} \ldots x_n^{a_n-1}\,.$$

$$y_1^{b_1-1} \ldots y_n^{b_n-1}\, e^{-(x+x_{k+1}+\ldots+x_n+y_1+\ldots+y_n)},$$

where $0 \leq x, x_{k+1},\ldots,x_n, y_1,\ldots,y_n \leq \infty$ and $Re(a) > 0$, $Re(a_i) > 0$, $i = k+1,\ldots,n$; $Re(b_j) > 0$, $j = 1,\ldots,n$. and $f = 0$ else where.

Consider another function

$$(9.3.11)\ g(x,x_{k+1},\ldots,x_n,y_1,\ldots,y_n) = {}_0F_1\left[-;c;\alpha_1yx+\ldots+\alpha_ky_kx + \alpha_{k+1}x_{k+1}y_{k+1} + \ldots + \alpha_nx_ny_n\right]$$

Now putting the values of f and g from (9.3.10) and (9.3.11) respectively in (9.2.1) and making an appeal to the result due to Chandel and Gupta [3,(5.5)], we obtain the following expectation of g having density function f

$$(9.3.12) \quad \langle g(x,x_{k+1},\ldots,x_n,y_1,\ldots,y_n)\rangle = {}^{(k)}F^{(n)}_{BD}\left[a,a_{k+1},\ldots,a_n,b_1,\ldots,b_n;\ c;\ \alpha_1,\ldots,\alpha_n\right].$$

We consider another density function

$$(9.3.13) \quad f(z,y,x_1,\ldots,x_k) = \frac{1}{\Gamma(a)\,\Gamma(b)\,\Gamma(b_1)\ldots\Gamma(b_k)}\, e^{-z-y-x_1-\ldots-x_n}\, z^{a-1}\, y^{b-1}\cdot x_1^{b_1-1}\ldots x_k^{b_k-1},$$

provided that $0 \le z,y,x_1,\ldots,x_k < \infty$ and $Re(a) > 0$, $Re(b) > 0$, $Re(b_i) > 0$, $i = 1,\ldots,k$ also $f = 0$, else where .

Further consider the function

$$(9.3.14) \quad g(z,y,x_1,\ldots,x_k) = {}_0F_1\left[-;c;zx_1\alpha_1+\ldots+zx_k\alpha_k\right]\ {}_0F_1\left[-;c_{k+1};zy\alpha_{k+1}\right]\ldots\ {}_0F_1\left[-;c_n;zy_n\alpha_n\right]$$

Now putting the values of f and g from (9.3.13) and (9.3.14) respectively in (9.2.1) and applying the result due to Chandel and Vishwakarma [4,p.178(3.9)], we derive corresponding Expectation of g

$$(9.3.15) \quad \langle g(z,v,x_1,\ldots,x_n)\rangle = {}^{(k)}F^{(n)}_{CD}[a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;\ \alpha_1,\ldots,\alpha_n]$$

where ${}^{(k)}F^{(n)}_{CD}$ is fourth possible intermediate Lauricella's multiple hypergeometric function due to Karlsson [9].

9.4 Expectations involving Confluent multiple hypergeometric functions related to multivariate Gamma distribution

Consider the density function

$$(9.4.1) \quad f(y,z) = \frac{1}{\Gamma(a)\,\Gamma(b)}\, y^{a-1}\, z^{b-1}\, e^{-(y+z)} ,$$

$Re(a) > 0$, $Re(b) > 0$, $0 \le y,z < \infty$ and $f(y,z) = 0$, else where .

Further consider

$$(9.4.2) \quad g(y,z) = {}_0F_1[-;c;y\alpha_1+\ldots+y\alpha_k]\ {}_0F_1[-;c_{k+1};yz\alpha_{k+1}]\ \ldots\ {}_0F_1[-;c_n;yz\alpha_n] .$$

The expectation for $g(y,z)$ corresponding to the function $f(y,z)$ is given by

$$(9.4.3) \quad \langle g(y,z)\rangle = {}^{(k)}_{(1)}\Phi^{(n)}_{CD}[a,b;c,c_{k+1},\ldots,c_n;\alpha_1,\ldots,\alpha_n] .$$

Consider the density function

$$(9.4.4) \quad f_1(y,x_1,\ldots,x_n) = \frac{1}{\Gamma(a)\ \Gamma(b_1)\ldots\ \Gamma(b_k)}\ e^{-(y+x_1+\ \ldots+x_k)}\ .$$

$$y^{a-1}\ .\ x_1^{\,b_1-1}\ \ldots\ x_k^{\,b_k-1}\ ,$$

$Re(a) > 0$, $Re(b_i) > 0$, $i=1,\ldots,k.$

All $0 \leq y,x_1,\ldots,x_k < \infty$ and $f_1(y,x_1,\ldots,x_k) = 0$, else where .

Further consider

$$(9.4.5)\ g(y,x_1,\ldots,x_k) = {}_0F_1\left[-;c;\alpha_1 x_1 y+\ldots+\alpha_k x_k y\right]\ {}_0F_1\left[-;c_{k+1};y\alpha_{k+1}\right]$$

$$\ldots\ {}_0F_1\left[-;c_n;y\alpha_n\right]\ ,$$

$0 \leq y < \infty$, $\alpha_{k+1},\ldots,\alpha_n$ are any real numbers , $c,c_{k+1},\ldots,c_n$ are neither zero nor negative integers .

Now making an appeal to the result due to Chandel and Vishwakarma [4,(3.11)] ,

we derive the expectation

$$(9.4.6)\ \langle g(y,x_1,\ldots,x_k)\rangle = {}^{(k)}_{(2)}\Phi^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;\alpha_1,\ldots,\alpha_n\right]$$

Now consider the density function

$$(9.4.7) \quad f_2(y,x_1,\ldots,x_k) = \frac{1}{\Gamma(a)\ \Gamma(b_1)\ \ldots \Gamma(b_k)} \cdot e^{-(y+x_1+\ldots+x_k)} \cdot y^{a-1}\ x_1^{b_1-1} \ldots x_k^{b_k-1} ,$$

$Re(a)>0$, $Re(b_i)>0$, $i=1,\ldots,k$; $0 \leq y, x_1, \ldots x_k < \infty$

$f_2(y,x_1,\ldots,x_k) = 0$, else where .

Take

$$(9.4.8) \quad g(y,x_1,\ldots,x_k) = {}_0F_1[-;c;\alpha_1 x_1+\ldots+\alpha_k x_k]\ {}_0F_1[-;c_{k+1};\alpha_{k+1}y] \ldots {}_0F_1[-;c_n;\alpha_n y] .$$

Now making an appeal to the result due to Chandel and Vishwakarma [4,(3.12)] , we derive the expectation

$$(9.4.9) \quad \langle g(y,x_1,\ldots,x_n)\rangle = {}^{(k)}_{(3)}\Gamma^{(n)}_{CD}[a,b,b_1,\ldots,b_k;c;c_{k+1},\ldots,c_n;\alpha_1,\ldots,\alpha_n] .$$

Consider the density function

$$(9.4.10) \quad f(y,z,x_1,\ldots,x_k) = \frac{1}{\Gamma(a)\Gamma(b)\ \Gamma(b_1)\ \ldots\ \Gamma(b_k)} \cdot e^{-(y+z+x_1+\ldots+x_k)} \cdot y^{a-1} \cdot z^{b-1} \cdot x_1^{b_1-1} \ldots x_k^{b_k-1} ,$$

$0 \leq y,z,x_1,\ldots,x_k < \infty$, $Re(a)>0$, $Re(b)>0$, $Re(b_i)>0$,

$i = 1,\ldots,k$ and $f(z,y,x_1,\ldots,x_k) = 0$, else where

and take corresponding function

$$(9.4.11) \quad g(y,z,x_1,\dots,x_k) = e^{yz(\alpha_{k+1}+\dots+\alpha_n)}\ {}_0F_1[-;c;y\alpha_1 x_1+\dots+y\alpha_k x_k]$$

where all $\alpha_i,(i=1,\dots,n)$ are any real numbers ,

Then the expectation for the function g with density function f is given by

$$(9.4.12) \quad \langle g(y,z,x_1,\dots,x_k)\rangle = {}^{(k)}_{(4)}\Phi^{(n)}_{CD}[a,b,b_1,\dots,b_k;c;\alpha_1,\dots,\alpha_n] .$$

Consider the density function

$$(9.4.13) \quad f(y,z) = \frac{1}{\Gamma(a)\ \Gamma(b)}\ e^{-(y+z)}\ y^{a-1}\ z^{b-1} ,$$

$Re(a)>0$, $Re(b)>0$, $0\le y,z\le\infty$ and $f(y,z)=0$, elsewhere .

and another function

$$(9.4.14) \quad g(y,z) = {}_0F_1[-;c_1;\alpha_1 yz]\dots{}_0F_1[-;c_k;\alpha_k yz]\cdot$$

$${}_0F_1[-;c_{k+1};\alpha_{k+1}y]\dots{}_0F_1[-;c_n;\alpha_n y] ,$$

$\alpha_1,\dots\alpha_n$ are any real numbers and $c_1,\dots,c_n$ are neither zero nor negative integers .

Thus the expectation of $g(y,z)$ having density function $f(y,z)$ is given by

$$(9.4.15) \quad \langle g(y,z)\rangle = {}^{(k)}_{(1)}\Phi^{(n)}_{AC}[a,b;c_1,\dots,c_n;\alpha_1,\dots,\alpha_n]$$

Consider the density function

$$(9.4.16) \qquad f(y) = \frac{1}{\Gamma(a)} \cdot e^{-y} \, y^{a-1} \,,$$

$Re(a) \geq 0$, $0 \leq y < \infty$, and $f(y) = 0$, else where .

Take another function

$$(9.4.17) \quad g(y) = {}_0F_1 [-; c_1; \alpha_1 y] \ldots {}_0F_1 [-; c_k; \alpha_k y] \cdot {}_1F_1 [b_{k+1}; c_{k+1}; \alpha_{k+1} y] \ldots {}_1F_1 [b_n; c_n; \alpha_n y] \,,$$

$0 \leq y < \infty$, $\alpha_1, \ldots, \alpha_n$ are any real numbers and $c_1, \ldots, c_n$ are neither zero nor negative integers,

Then the expectation of $g(y)$ having density function $f(y)$ is given by

$$(9.4.18) \quad < g(y) > = {}^{(k)}_{(2)}\Phi^{(n)}_{AC} [a, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; \alpha_1, \ldots, \alpha_n]$$

Consider the probability density function

$$(9.4.19) \qquad f(y,z) = \frac{\Gamma(c)}{2\pi i \Gamma(a)} \cdot e^{-y+z} \, y^{a-1} \, z^{-c} \,,$$

$Re(a) > 0$, $Re(c) > 0$, $b_1, \ldots, b_k$ are negative integers

$0 \leq y < \infty$, $-\infty \leq z \leq 0$ and $f(y,z) = 0$, else where .

Consider another function

$$(9.4.20)\quad g(y,z) = \left(1-\frac{\alpha_1 y}{z}\right)^{-b_1} \cdots \left(1-\frac{\alpha_k y}{z}\right)^{-b_k} (1-\alpha_{k+1} z)^{-b_{k+1}} \cdots (1-\alpha_n z)^{-b_n},$$

$\alpha_1, \ldots, \alpha_n$ are real numbers, $b_{k+1}, \ldots, b_n$ are negative integers.

Then the expectation for $g(y,z)$ having density function $f(y,z)$ is given by

$$(9.4.21)\quad \langle g(y,z)\rangle = {}^{(k)}_{(1)}\Phi^{(n)}_{AD}\left[a, b_1, \ldots, b_n; c; \alpha_1, \ldots \alpha_n\right].$$

Consider the density function

$$(9.4.22)\quad f(y,z_1,\ldots,z_n) = \frac{1}{\Gamma(a)\,\Gamma(b_1)\ldots\Gamma(b_n)} \cdot e^{-(y+z_1+\ldots+z_n)} \cdot y^{a-1}\, z_1^{b_1-1} \ldots z_n^{b_n-1},$$

$Re(a) > 0$, $Re(b_i) > 0$, $i = 1, \ldots, n$; $0 \leq y, z_1, \ldots, z_n < \infty$ and $f(y,z_1,\ldots,z_n) = 0$, else where.

Take

$$(9.4.23)\quad g(y,z_1,\ldots,z_n) = {}_0F_1\left[-; c; \alpha_1 z_1 y + \ldots + \alpha_k z_k y + \alpha_{k+1} z_{k+1} + \ldots + \alpha_n z_n\right].$$

Thus the expectation for $f(y,z_1,\ldots,z_n)$ having density function $g(y,z_1,\ldots,z_n)$ is given by

$$(9.4.24) < g(y,z_1,\ldots,z_n) > = {}^{(k)}_{(1)}\Phi^{(n)}_{BD} [a, b_1, \ldots, b_n; c; \alpha_1, \ldots, \alpha_n] .$$

Consider the density function

$$(9.4.25) \quad f(z_1,\ldots,z_n,y_{k+1},\ldots,y_n) = \frac{1}{\Gamma(a_{k+1}) \ldots \Gamma(a_n)\, \Gamma(b_1) \ldots \Gamma(b_n)} .$$

$$\exp[-(z_1+\ldots+z_n+y_{k+1}+\ldots+y_n)]\, y_{k+1}^{a_{k+1}-1} \ldots y_n^{a_n-1} \cdot z_1^{b_1-1} \ldots z_n^{b_n-1} ,$$

$Re(a_i) > 0$, $Re(b_j) > 0$, $i = k+1,\ldots,n$; $j = 1, \ldots, n$;

$0 \leq z_1,\ldots,z_n, y_{k+1},\ldots,y_n < \infty$ and $y(z_1,\ldots,z_n,y_{k+1},\ldots,y_n) = 0$

else where .

Take

$$(9.4.26) \quad g(z_1,\ldots,z_n,y_{k+1},\ldots,y_n) = {}_0F_1[-;c;\alpha_1 z_1+\ldots+\alpha_k z_k+\alpha_{k+1} y_{k+1} z_{k+1}+ \ldots+\alpha_n y_n z_n] .$$

Thus the expectation for $f(z_1,\ldots,z_n,y_{k+1},\ldots,y_n)$ having density function $g(z_1,\ldots,z_n,y_{k+1},\ldots,y_n)$ is given by

$$(9.4.27) < g(z_1,\ldots,z_n,y_{k+1},\ldots,y_n) > = {}^{(k)}_{(2)}\Phi^{(n)}_{BD} [a_{k+1},\ldots,a_n, b_1,\ldots,b_n; c; \alpha_1,\ldots,\alpha_n]$$

Consider the density function

$$(9.4.28) \quad f(x,y) = \frac{1}{\Gamma(a)\, \Gamma(b)} \cdot e^{-(x+y)}\, x^{a-1}\, y^{b-1} ,$$

$Re(a) > 0$, $Re(b) > 0$, $0 \leq x, y, < \infty$ and $f(x,y) = 0$, else where.

Take

$$(9.4.29) \quad g(x,y) = (1-\alpha_1 x)^{-b_1} \ldots (1-\alpha_k x)^{-b_k} \; {}_0F_1[-; c_{k+1}; \alpha_k\, xy]$$

$$\ldots \; {}_0F_1[-; c_n; \alpha_n xy]$$

$0 \leq x, y < \infty$, $\alpha_1, \ldots, \alpha_n$ are any real numbers and $c_{k+1}, \ldots, c_n$ are neither zero nor negative integers.

Thus the expectation for the function $g(x,y)$ having density function $f(x,y)$ is given by

$$(9.4.30) \quad \langle g(x,y) \rangle = \frac{(k)}{(5)}\Gamma(n)\,\phi_{CD}[a, b, b_1, \ldots, b_k; c_{k+1}, \ldots, c_n; \alpha_1, \ldots, \alpha_n].$$

Consider the density function

$$(9.4.31) \quad f(x, y, y_1, \ldots, y_k) = e^{-(x+y+y_1+\ldots+y_k)} \cdot x^{a-1} \cdot y^{b-1} \cdot y_1^{b_1-1} \ldots y_k^{b_k-1},$$

$Re(a) > 0$, $Re(b) > 0$, $Re(b_j) > 0$, $j = 1, \ldots, k$; $0 \leq x, y, y_1, \ldots, y_k < \infty$

and $f(x, y, y_1, \ldots, y_k) = 0$ else where .

Take

$$(9.4.32) \quad g(x, y, y_1, \ldots, y_k) = e^{x(y_1\alpha_1 + \ldots + y_k\alpha_k)} \cdot {}_0F_1[-; c_{k+1}; xy\alpha_{k+1}]$$

$$\ldots \; {}_0F_1[-; c_n; xy\,\alpha_n] ,$$

$0 \leq x, y \leq \infty$, $c_{k+1}, \ldots, c_n$ are neither zero nor negative intigers while $\alpha_1, \ldots, \alpha_n$ are any real numbers .

Thus the expectation for the function $g(x,y)$ having density function $f(x,y,y_1,\ldots,y_k)$ is given by

$$(9.4.33) \quad \langle g(x,y)\rangle = {}^{(k)}_{(5)}I^{(n)}_{CD}\left[a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;\alpha_1,\ldots,\alpha_n\right],$$

Consider the density function $f(x)$

$$(9.4.34) \quad f(x) = \frac{1}{\Gamma(a)}\cdot e^{-x}\,x^{a-1} \quad , \quad Re(a)>0 \quad , \quad 0 \leq x \leq \infty .$$

Further consider

$$(9.4.35) \quad g(x) = (1-\alpha_1 x)^{-b_1}\ldots(1-\alpha_k x)^{-b_k}\;{}_0F_1\left[-;c_{k+1};x\alpha_{k+1}\right] \ldots {}_0F_1\left[-;c_n;\,x\alpha_n\right],$$

$c_{k+1},\ldots,c_n$ are neither zero nor negative integers and $\alpha_1,\ldots\alpha_n$ are any real numbers.

Thus the expectation for the function $g(x)$ having density function $f(x)$ is given by

$$(9.4.36) \quad \langle g(x)\rangle = {}^{(k)}_{(6)}I^{(n)}_{CD}\left[a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;\;\alpha_1,\ldots,\alpha_n\right].$$

Consider the density function

$$(9.4.37)\quad f(x,y_1,\dots,y_k) = \frac{1}{\Gamma(a)\,\Gamma(b_1)\dots\Gamma(b_k)}\, x^{a-1}\, y_1^{b_1-1} \dots y_k^{b_k-1}$$

$Re(a)>0$, $Re(b_j)>0$, $j = 1,\dots,k$, $0 \le x, y_1,\dots,y_k < \infty$

and $f(x,y_1,\dots,y_k) = 0$, else where .

Take

$$(9.4.38)\, g(x,y_1,\dots,y_k) = \exp[x(y_1\alpha_1+\dots+y_k\alpha_k)] \cdot {}_0F_1[-;c_{k+1};\alpha_{k+1}x] \dots {}_0F_1[-;c_n;\alpha_n x],$$

$c_{k+1},\dots,c_n$ are neither zero nor negative integers, while

$\alpha_1,\dots,\alpha_n$ are any real numbers.

Thus the expectation for the function $g(x,y_1,\dots,y_k)$ having

density function $f(x,y_1,\dots,y_k)$ is given by

$$(9.4.39)\langle g(x,y_1,\dots,y_k)\rangle = {}^{(k)}_{(6)}I^{(n)}_{CD}[a,b_1,\dots,b_k;c_{k+1},\dots,c_n;\alpha_1,\dots,\alpha_n].$$

Consider the density function

$$(9.4.40)\quad f(x) = \frac{1}{\Gamma(a)} \cdot e^{-x}\, x^{a-1},$$

$Re(a)>0$, $0 \le x < \infty$ and $f(x) = 0$, else where .

Further Consider

$$(9.4.41)\quad g(x) = \prod_{j=1}^{k} (1-\alpha_j x)^{-b_j}\; {}_1F_1[b_{k+1};c_{k+1};\alpha_{k+1}x] \dots$$

$$, {}_1F_1[b_n;c_n;\alpha_n x]$$

b_i, α_i, $\{i=1,\ldots,n\}$ are any real numbers and $c_{k+1},\ldots,c_n$ are neither zero nor negative integers

Then the expectation for $g(x)$ having density function $f(x)$ is given by

$$(9.4.42) \quad \langle g(x)\rangle = {}^{(k\ )}_{(2)}\Phi^{(n)}_{AD}\left[a, b_1,\ldots,b_n; c_{k+1},\ldots,c_n; \alpha_1,\ldots,\alpha_n\right].$$

Consider the density function

$$(9.4.43) \quad f(x,y_1,\ldots,y_n) = \frac{1}{\Gamma(a)\ \Gamma(b_1)\ \ldots\Gamma(b_n)}\, e^{-(x+y_1+\ldots+y_n)}\,.$$

$$x^{a-1}\ y_1^{b_1-1}\ \ldots\ y_n^{b_n-1}\ ,$$

$Re(a) > 0$, $Re(b_j) > 0$; $j = 1,\ldots, n$, $0 \le x, y_1,\ldots,y_k \le \infty$,

$\alpha_1,\ldots,\alpha_k$ are any real numbers and $f(x,y_1,\ldots,y_n) = 0$, else where .

$$(9.4.44) \quad g(x,y_1,\ldots,y_n) = e^{x(\alpha_1 y_1+\ldots+\alpha_k y_k)}\ {}_0F_1\left[-;c_{k+1};\alpha_{k+1}y_{k+1}x\right]$$

$$\ldots\ {}_0F_1\left[-;c_n;\alpha_n y_n x\right]$$

where $c_j(j=k+1,\ldots,n)$ are neither zero nor negative integers while $c_j(j=1,\ldots,n)$ are any real numbers.

Then the expectation for $g(x,y_1,\ldots,y_n)$ having density function $f(x,y_1,\ldots,y_n)$ is given by

(9.4.45) $\langle g(x,y_{k+1},\dots,y_n)\rangle = {}^{(k)}_{(2)}\Phi^{(n)}_{AD}\left[a,b_1,\dots,b_n;c_{k+1},\dots,c_n;\alpha_1,\dots,\alpha_n\right]$

Consider the density function

(9.4.46) $$f(x,y_1,\dots,y_k,x_{k+1},\dots,x_n) = \frac{1}{\Gamma(a)\,\Gamma(b_1)\dots\Gamma(b_k)\,\Gamma(a_{k+1})\dots\Gamma(a_n)}\cdot$$

$$e^{-(x+y_1+\dots+y_k+x_{k+1}+\dots+x_n)}\cdot x^{a-1}\cdot\prod_{j=1}^{k} y_i^{b_i-1}\prod_{j=k+1}^{n} x_j^{a_j-1},$$

$Re(a)>0$, $Re(b_i)>0$, $i=1,\dots,k$, $Re(a_j)>0$, $j=k+1,\dots,n$.

$0\le x$, $y_1,\dots,y_k,x_{k+1},\dots,x_n<\infty$ and $f(x,y_1,\dots,y_k,x_{k+1},\dots,x_n)=0$, else where .

Take

(9.4.47) $$g(x,y_1,\dots,y_k,x_{k+1},\dots,x_n) = {}_0F_1\left[-;c;\alpha_1xy_1+\dots+\alpha_kxy_k+\alpha_{k+1}x_{k+1}+\dots+\alpha_nx_n\right],$$

$0\le x,y_1,\dots,y_k,x_{k+1},\dots,x_n\le\infty$, $\alpha_1,\dots,\alpha_n$ are any real numbers where c is neither zero nor negative integers .

Then the expectation for $g(x,y_1,\dots,y_k,x_{k+1},\dots,x_n)$ having density function $f(x,y_1,\dots,y_k,x_{k+1},\dots,x_n)$ is given by

(9.4.48) $$\langle g(x,y_1,\dots,y_k,x_{k+1},\dots,x_n)\rangle =$$

$${}^{(k)}_{(3)}\Phi^{(n)}_{BD}\left[a,a_{k+1},\dots,a_n,b_1,\dots,b_k;c;\alpha_1,\dots,\alpha_n\right].$$

Consider the density function

$$(9.4.49) \quad f(x,y_1,\ldots,y_n) = x^{a-1}\, y_1^{b_1-1} \cdots y_n^{b_n-1}\, e^{-(y_1+\ldots+y_n+x)}$$

$Re(a)>0$, $Re(b_i)>0$, $i = 1, \ldots, n$ $0 \le x, y_1, \ldots, y_n < \infty$ and

$f(x,y_1,\ldots,y_n) = 0$, else where .

Take

$$(9.4.50) \quad g(x,y_1,\ldots,y_n) = e^{x(\alpha_{k+1} y_{k+1} + \cdots + \alpha_n y_n)}\ {}_0F_1[-; c; \alpha_1 y_1 x, \ldots, \alpha_k y_k x],$$

where $\alpha_1, \ldots, \alpha_n$ are any real numbers while c is neither zero nor negative integer .

Then the expectation for the function $g(x,y_1,\ldots,y_n)$ having density function $f(x,y_1,\ldots,y_n)$ is given by

$$(9.4.51) \quad \langle g(x,y_1,\ldots,y_n)\rangle = {}^{(k)}_{(1)}\Phi^{(n)}_D [a, b_1, \ldots, b_n; c; \alpha_1, \ldots, \alpha_n]$$

Consider the density function

$$(9.4.52) \quad f(x,y) = \frac{1}{\Gamma(a)\,\Gamma(b)}\, e^{-(x+y)}\, x^{a-1}\, y^{b-1} ,$$

$Re(a)>0$, $Re(b)>0$, $0 \le x, y \le \infty$, and $f(x,y) = 0$, else where .

Take

$$(9.4.53) \quad g(x,y) = {}_0F_1[-; c_1; xy\alpha_1] \cdots {}_0F_1[-; c_k; xy\alpha_k] \cdot$$

$$ {}_0F_1[-; c_{k+1}; y\alpha_{k+1}] \cdots {}_0F_1[-; c_n; y\alpha_n] ,$$

$\alpha_1,\ldots,\alpha_n$ are any real numbers while $c_1,\ldots,c_n$ are neither zero nor negative integers.

Then the expectation for the function $g(x,y)$ having density function $f(x,y)$ is given by

$$(9.4.54)\quad \langle g(x,y)\rangle = {}^{(k)}_{(1)}\Phi^{(n)}_{C}\left[a,b;c_1,\ldots,c_n;x_1,\ldots,x_n\right].$$

Consider the density function

$$(9.4.55)\quad f(x,y_1,\ldots,y_n) = \frac{1}{\Gamma(a)\,\Gamma(b_1)\ldots\Gamma(b_n)}\cdot e^{-(x+y_1+\ldots+y_n)}\, x^{a-1}\,.\; y_1^{b_1-1}\ldots y_n^{b_n-1},$$

$Re(a)>0$, $Re(b_j)>0$, $(j=1,\ldots,n)$; $0\le x,y_1,\ldots,y_n<\infty$ and

$f(x,y_1,\ldots,y_n) = 0$, else where .

Take

$$(9.4.56)\quad g(x,y_1,\ldots,y_n) = {}_0F_1\left[-;c;xy_1\alpha_1+\ldots+xy_k\alpha_k+y_{k+1}\alpha_{k+1}+\ldots+y_n\alpha_n\right],$$

where c is neither zero nor negative integer.

Then the expectation for the function $g(x,y_1,\ldots,y_n)$ having density function $f(x,y_1,\ldots,y_n)$ is given by

$$(9.4.57)\quad \langle g(x,y_1,\ldots,y_n)\rangle = {}^{(k)}_{(2)}\Phi^{(n)}_{D}\left[a,b_1,\ldots b_n;c;x_1,\ldots,x_n\right].$$

9.5 Special Cases

(9.5.1) For k=0, from (9.2.5) and (9.2.9), we deduce eparately the result due to Exton [7,(7.2.1.5)] .

(9.5.2) For k=0, from (9.2.6), we derive the result due to Exton [7,(7.2.3.1)] ,

while

(9.5.3) For k=n, from (9.2.6), we obtain the result due to Exton [7,(7.2.1.5)].

(9.5.4) For k=0, (9.2.11) gives the result due to Exton [7,(2.2.1.6)],

while

(9.5.5) For k=n, (9.2.11) gives the result due to Exton [7,(7.2.1.5)].

(9.5.6) Further for k=0, from (9.3.6) and (9.3.9), we deduce the result due to Exton [7,(7.2.3.2)].

Also

(9.5.7) For k=n, from (9.3 .12) and (9.3.15), we derive the result due to Exton [7(7.2.3.2)].

For further special interest, we have also obtained expectations in terms of hypergeometric functions of four variables of Exton [6] and Sharma and Parihar [12], but due to lack of space, we do not produce them here.

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